Complex actions, complex Langevin and Lefschetz thimbles

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Motivation

QCD partition function

at nonzero quark chemical potential

$$[\det D(\mu)]^* = \det D(-\mu^*)$$

- fermion determinant is complex
- straightforward importance sampling not possible
- sign problem

$\Rightarrow$ QCD phase diagram has not yet been determined non-perturbatively

Outline

- complex actions
- Langevin dynamics
- thimble dynamics
- Langevin versus Lefschetz
- summary

GA, Phys. Rev. D 88 (2013) 094501 (1308.4811)
GA, Lorenzo Bongiovanni, Erhard Seiler & Dénes Sexty, 1407.2090

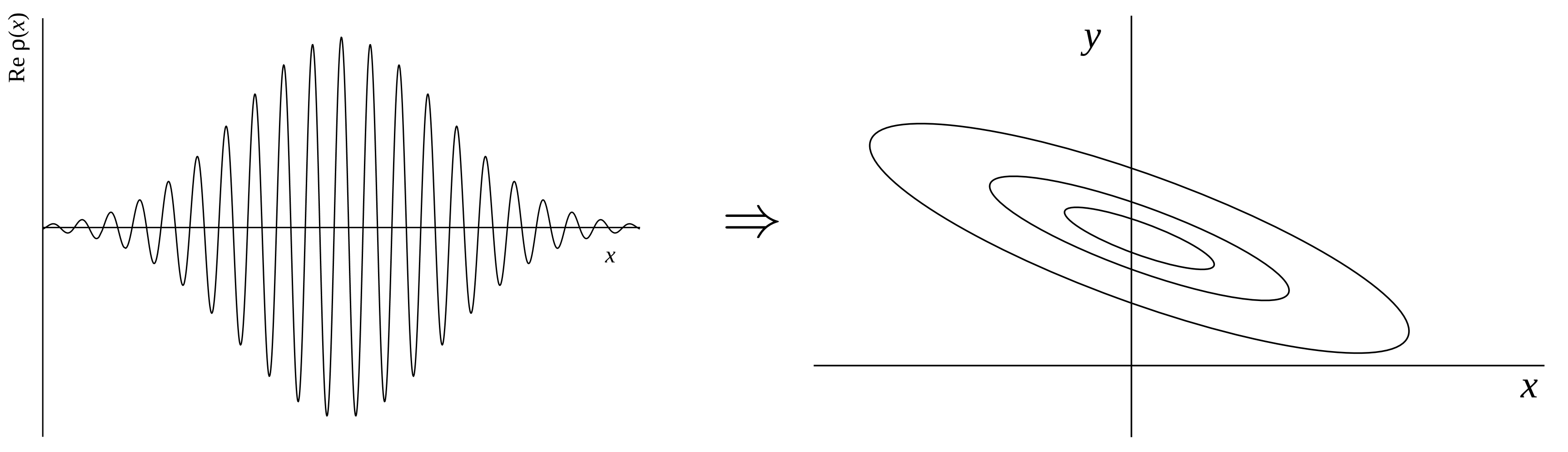
GA, Pietro Giudice & Erhard Seiler, Annals Phys. 337 (2013) 238 (1306.3075)

Complex actions

one degree of freedom: $$Z = \int dx\, e^{-S(x)}$$

complex holomorphic action $S(z) \in \mathbb{C}$

- numerical sign problem
- dominant configurations in the (path) integral?



- real and positive distribution $P(x, y)$?

Complex actions

various approaches relying on holomorphicity:

go into the complex plane

- saddle point/steepest descent: Lefschetz thimbles

  Witten 10
  Cristoforetti, Di Renzo, Mukherjee, Scorzato, (Schmidt) 12-14
  Fujii, Honda, Kato, Kikukawa, Komatsu, Sano 13
  Dunne, Unsal et al 12-14
  ...

- complex Langevin dynamics/stochastic quantisation

  GA, Seiler, Sexty, Stamatescu,
  James, Bongiovanni, Giudice, Jaeger, Attanasio
  ...

see talk by Dénes Sexty for progress in gauge theories

Complex Langevin dynamics

Langevin dynamics:

zero-dimensional example
complex action $S(z)$

- $\dot{z} = -\partial_z S(z) + \eta \qquad\qquad z = x + iy$
- associated Fokker-Planck equation (FPE)

$$\dot{P}(x,y;t) = [\partial_x(\partial_x + \mathrm{Re}\partial_z S(z)) + \partial_y \mathrm{Im}\partial_z S(z)]P(x,y;t)$$

- (equilibrium) distribution in complex plane: $P(x,y)$
- observables

$$\langle O(x+iy)\rangle = \frac{\int dxdy\, P(x,y) O(x+iy)}{\int dxdy\, P(x,y)}$$

- $P(x,y)$ real and non-negative: no sign problem
- criteria for correctness

Lefschetz thimbles

generalised saddle point integration/steepest descent:

extend definition of path integral Witten 10

- Chern-Simons theories
- mathematical foundation in Morse theory

formulation:

- find *all* stationary points z_k of holomorphic action $S(z)$
- paths of steepest descent: stable thimbles $\mathcal{J}_k$
- paths of steepest ascent: unstable thimbles $\mathcal{K}_k$
- $\mathrm{Im}\, S(z)$ constant along thimble k

integrate over stable thimbles, with proper weighting

Lefschetz thimbles

generalised saddle point integration/steepest descent:

- integrate over stable thimbles

$$Z = \sum_k m_k e^{-i\mathrm{Im}S(z_k)} \int_{\mathcal{J}_k} dz\, e^{-\mathrm{Re}S(z)}$$
$$= \sum_k m_k e^{-i\mathrm{Im}S(z_k)} \int ds\, z'(s) e^{-\mathrm{Re}S(z(s))}$$

- intersection numbers: $m_k = \langle C, \mathcal{K}_k \rangle$
(${C} =$ original contour, $\mathcal{K}_k =$ unstable thimble)
- residual sign problem: complex Jacobian $J(s) = z'(s)$
- global sign problem: phases $e^{-i\mathrm{Im}S(z_k)}$

Lefschetz thimbles

numerical Lefschetz approach di Renzo et al 12

- find all saddle points/thimbles in field theory?
- integrate over dominant thimble $\mathcal{J}_0$ only

$$Z = e^{-i\mathrm{Im}S(z_0)} \int_{\mathcal{J}_0} dz \, e^{-\mathrm{Re}S(z)}$$

- motivated by universality
- no global sign problem
- residual sign problem remaining

validity?

- successful e.g. in interacting 4-dim Bose gas with $\mu \neq 0$

Langevin versus Lefschetz

two approaches in the complex plane:

- Langevin

$$\langle O(z) \rangle = \frac{\int dxdy\, P(x,y) O(x+iy)}{\int dxdy\, P(x,y)}$$

- Lefschetz

$$\langle O(z) \rangle = \frac{\sum_k m_k e^{-i\mathrm{Im}S(z_k)} \int_{\mathcal{J}_k} dz\, e^{-\mathrm{Re}S(z)} O(z)}{\sum_k m_k e^{-i\mathrm{Im}S(z_k)} \int_{\mathcal{J}_k} dz\, e^{-\mathrm{Re}S(z)}}$$

- two- versus one-dimensional
- real versus residual/global phases

relation? validity? $\Rightarrow$ simple models

Langevin versus Lefschetz

Quartic model

$$Z = \int_{-\infty}^{\infty} dx\, e^{-S} \qquad\qquad S(x) = \frac{\sigma}{2}x^2 + \frac{\lambda}{4}x^4$$

complex mass parameter $\sigma = A + iB$, $\lambda \in \mathbb{R}$

often used toy model Ambjorn & Yang 85, Klauder & Petersen 85, Okamoto et al 89, Duncan & Niedermaier 12

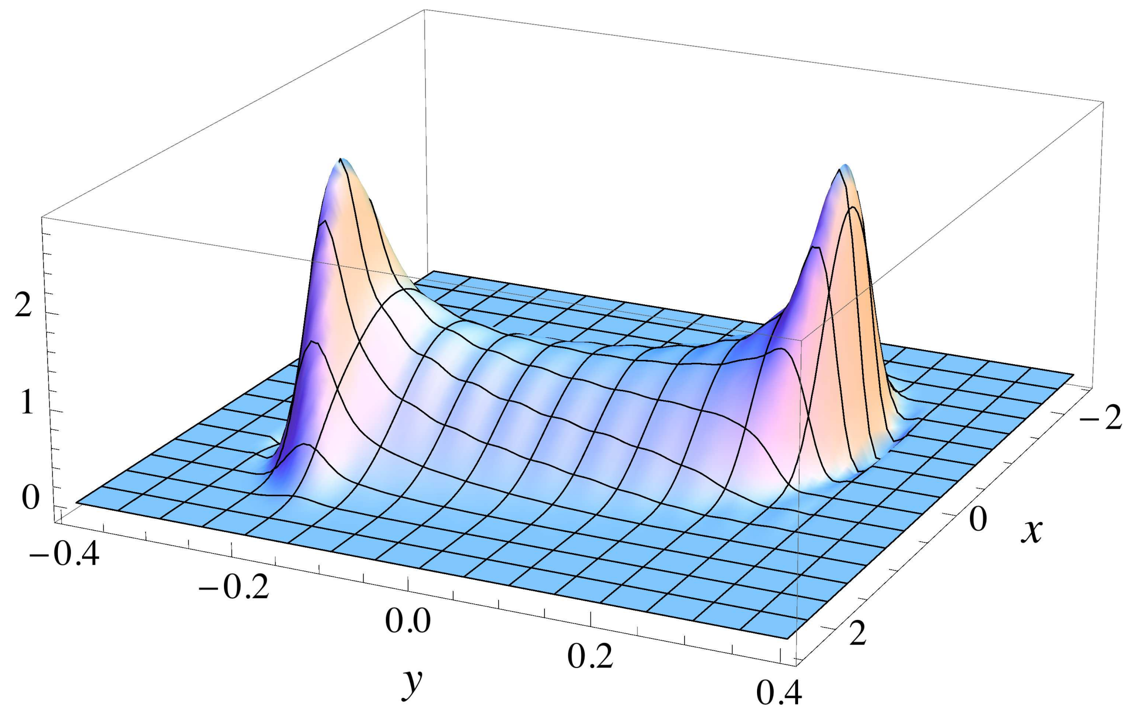
essentially analytical proof for CL*: GA, Giudice & Seiler 13

- CL gives correct result for all observables $\langle x^n \rangle$ provided that $A > 0$ and $A^2 > B^2/3$
- based on properties of the distribution $P(x, y)$
- follows from classical flow or directly from FPE

* GA, Seiler, Stamatescu 09 + James 11

Quartic model

- numerical solution of FPE for $P(x, y)$
- distribution is localised in a strip around real axis
- $P(x, y) = 0$ when $|y| > y_-$ with $y_- = 0.303$ for $\sigma = 1 + i$

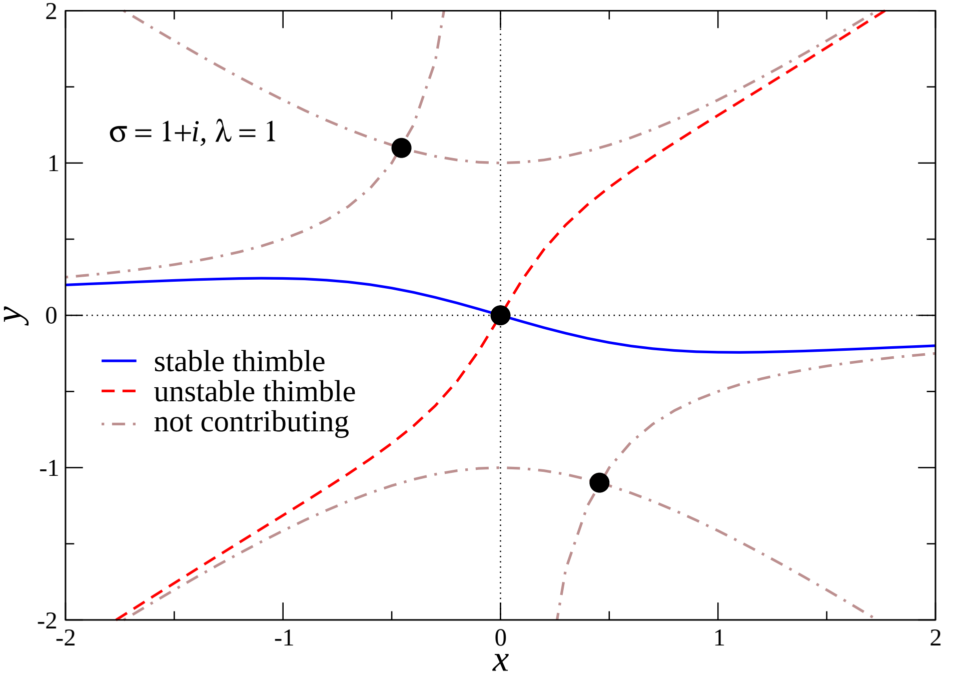


Langevin versus Lefschetz

Lefschetz thimbles for quartic model

- critical points:

$$z_0 = 0$$
$$z_\pm = \pm i\sqrt{\sigma/\lambda}$$



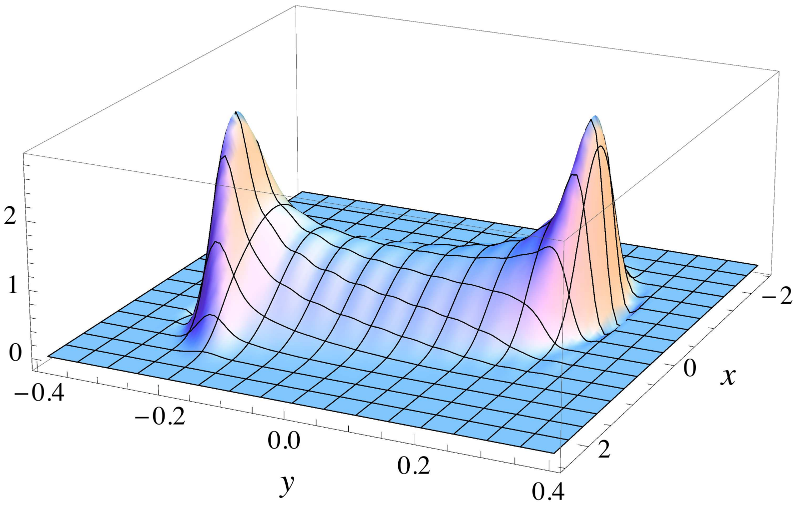
- thimbles can be computed analytically

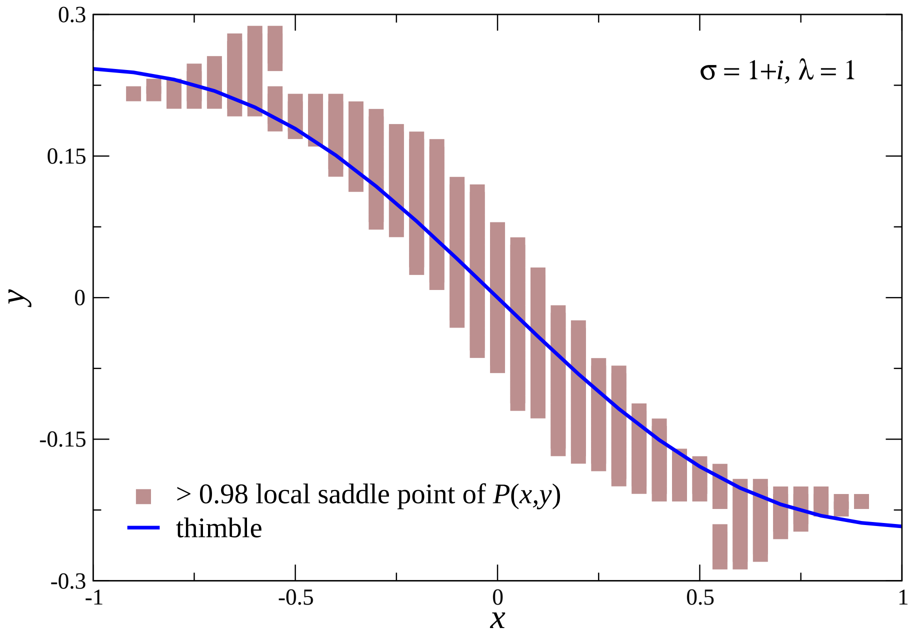
$$\mathrm{Im}S(z_0) = 0$$
$$\mathrm{Im}S(z_\pm) = -AB/2\lambda$$

- for $A > 0$: only 1 thimble contributes
- integrating along thimble gives correct result, with inclusion of complex Jacobian

Quartic model: thimbles

compare thimble and FP distribution $P(x, y)$

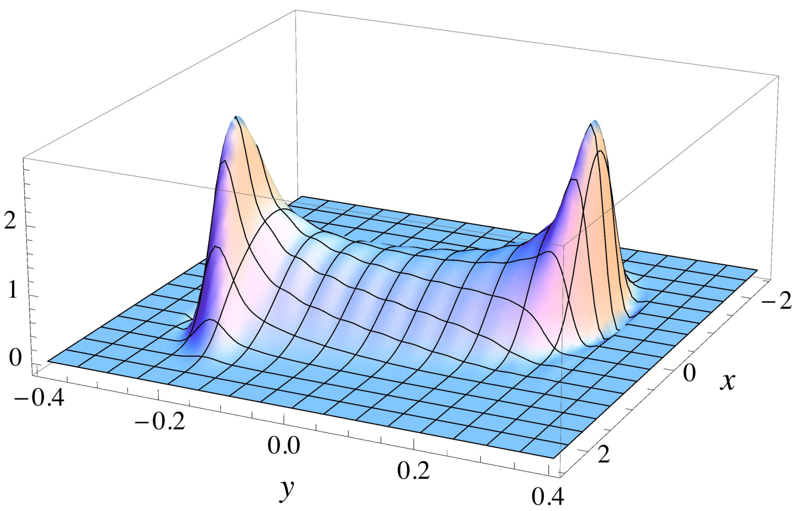


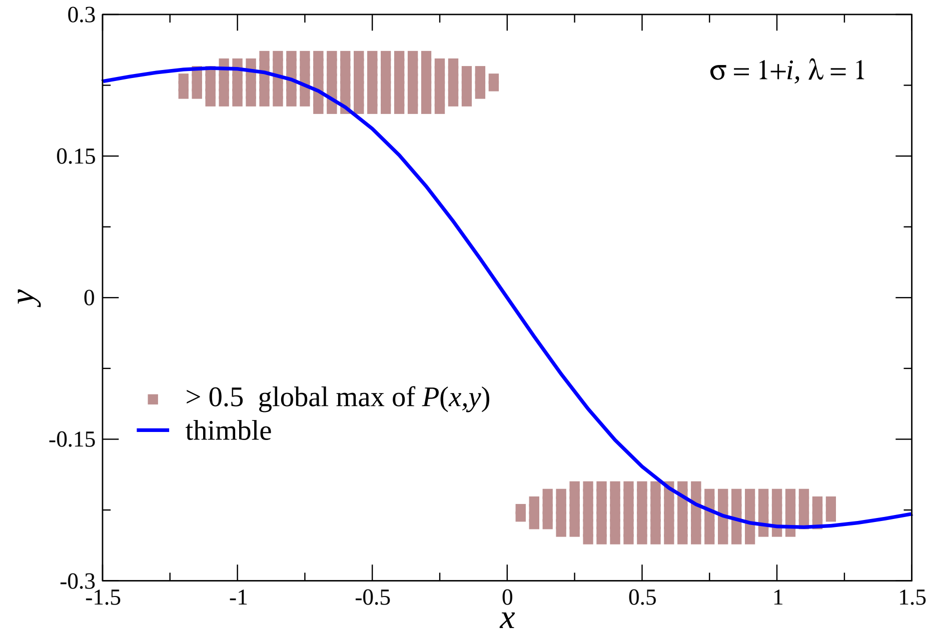


- thimble and $P(x, y)$ follow each other

Quartic model: thimbles

compare thimble and FP distribution $P(x, y)$





- thimble and $P(x, y)$ follow each other
- however, weight distribution quite different

intriguing result: complex Langevin process finds the thimble – is this generic?

Langevin versus Lefschetz

compare evolution equations in more detail

- complex Langevin (CL) dynamics

$$\dot{x} = -\mathrm{Re}\,\partial_z S(z) + \eta \qquad \dot{y} = -\mathrm{Im}\,\partial_z S(z)$$

- Lefshetz thimble dynamics, with $z(t \to \infty) = z_k$

$$\dot{x} = -\mathrm{Re}\,\partial_z S(z) \qquad \dot{y} = +\mathrm{Im}\,\partial_z S(z)$$

$\Rightarrow$ change in sign for y drift

Langevin:

- stable and unstable fixed points
- unstable runaways as $y \to \pm\infty$

thimbles:

- saddle points
- stable thimbles coming from $y \to \pm\infty$

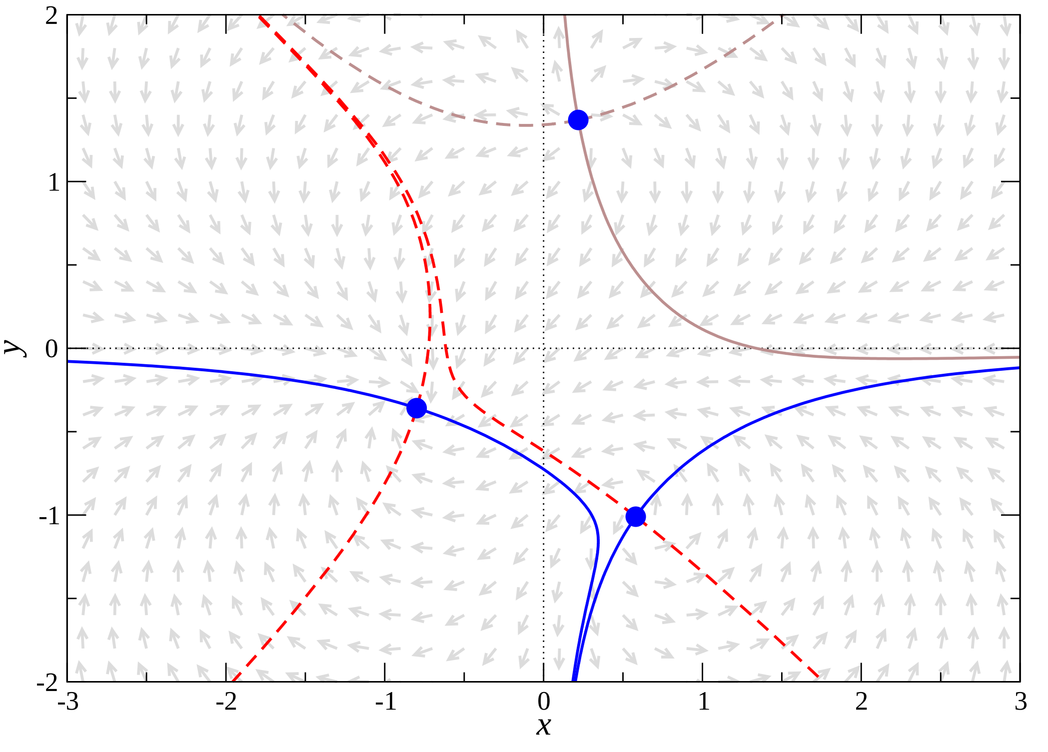
Langevin versus Lefschetz

deform quartic model with linear term, break symmetry

$$S(z) = \frac{\sigma}{2}z^2 + \frac{1}{4}z^4 + hz$$

$$h \in \mathbb{C}$$

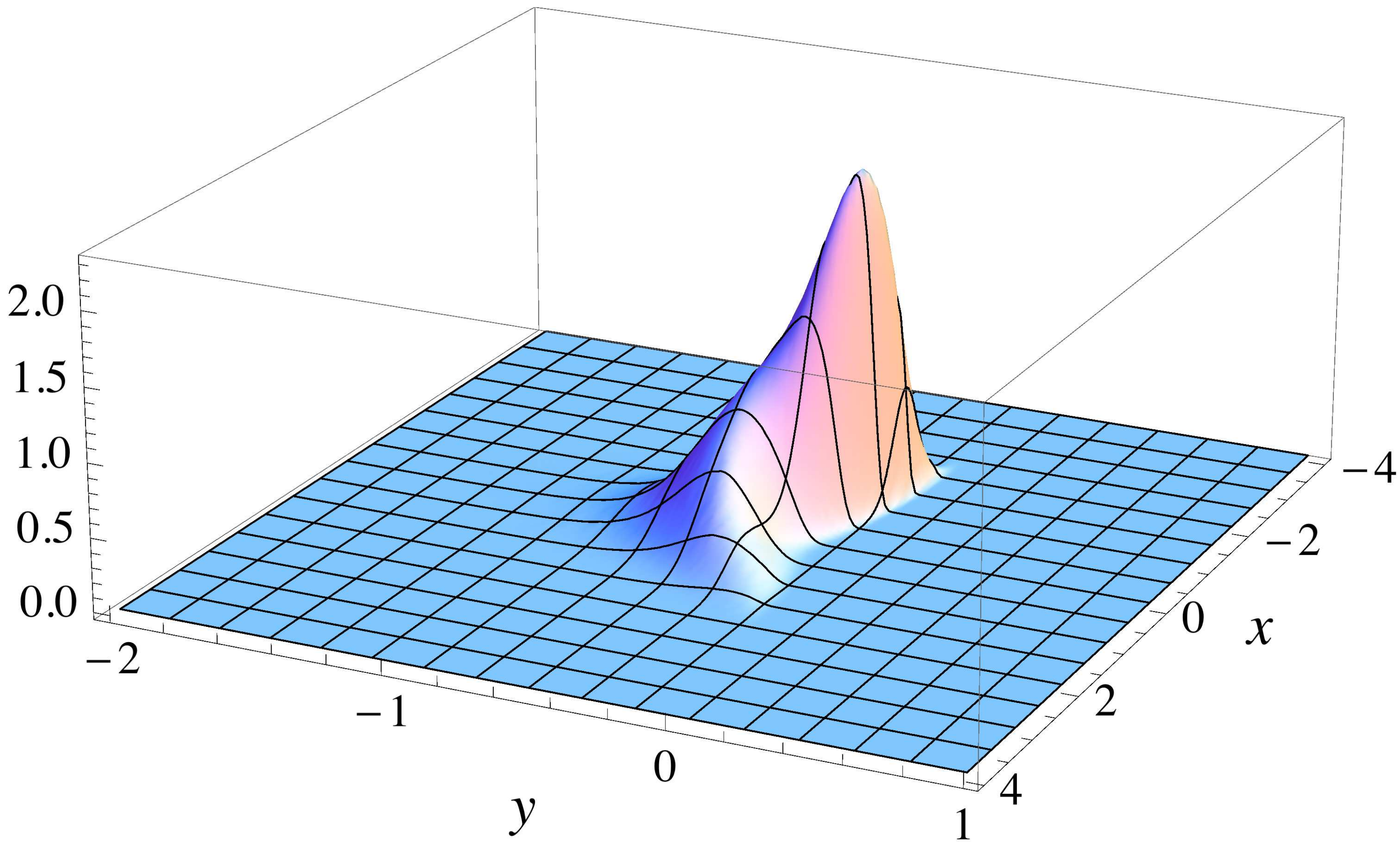
Langevin flow for
$\sigma = 1, h = 1 + i$



- one stable/two unstable fixed points for CL
- $y \to -\infty$ classical runaway trajectory
- two contributing thimbles (global phase problem) due to Stokes' phenomenon (1847)

Langevin versus Lefschetz

histogram of $P(x, y)$ collected during CL simulation

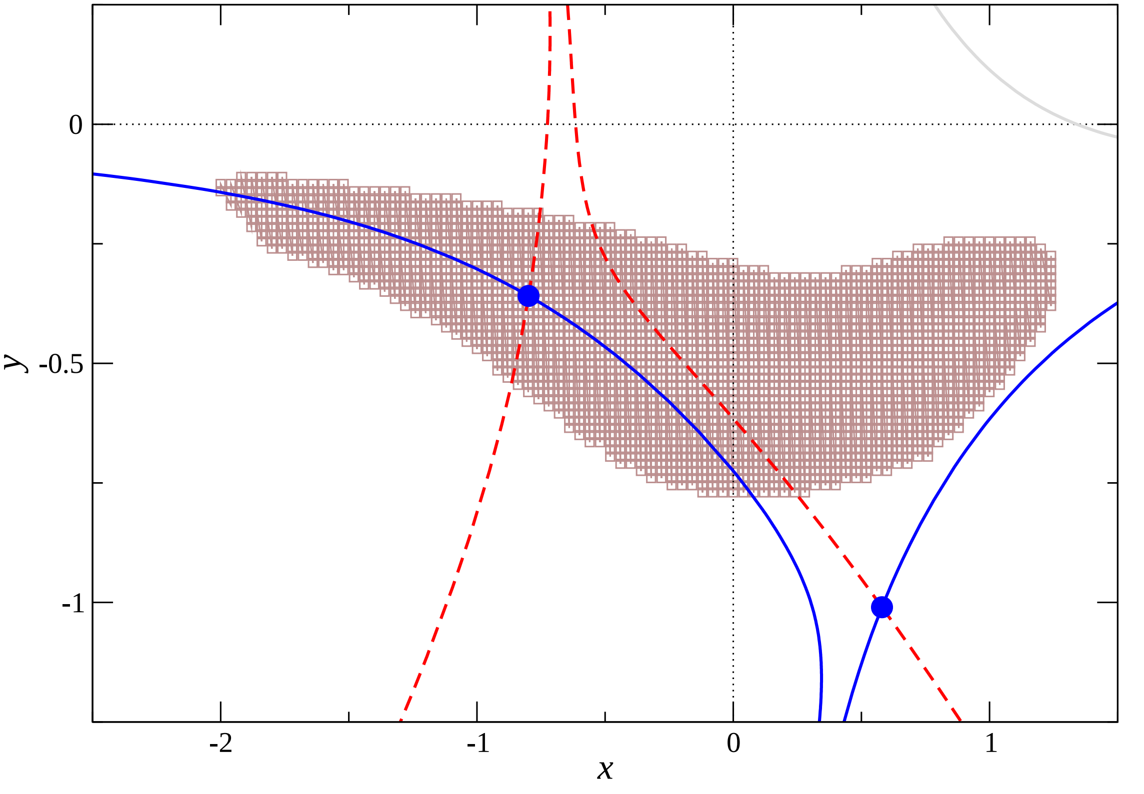


Langevin versus Lefschetz

comparison of

Langevin distribution

with thimbles



- thimbles: both saddle points contribute
- CL: unstable fixed point avoided
- no role for second thimble in Langevin

$\Rightarrow$ distributions manifestly different

Other (more relevant) models

U(1) model with determinant

$$Z = \int_{-\pi}^{\pi} dx\, e^{\beta \cos x}[1 + \kappa \cos(x - i\mu)]$$

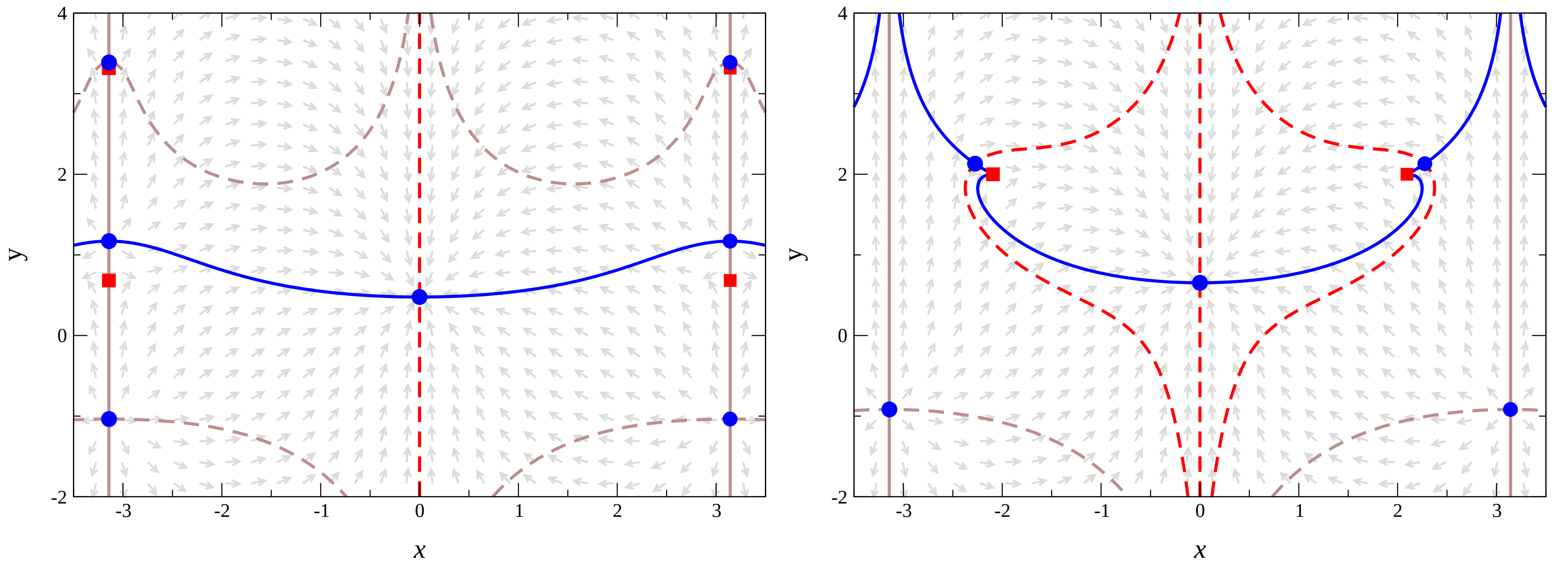
- presence of $\log \det$ of interest for CL and thimbles
  GA & Stamatescu 08, Mollgaard & Splittorff 13, Greensite 14

SU(2) one-link model with complex β

$$Z = \int dU\, e^{-S(U)} \qquad\qquad S(U) = -\frac{\beta}{2}\mathrm{Tr}\, U$$

- solvable with CL in different ways (gauge fixing, gauge cooling, $\ldots$)

U(1) model with determinant

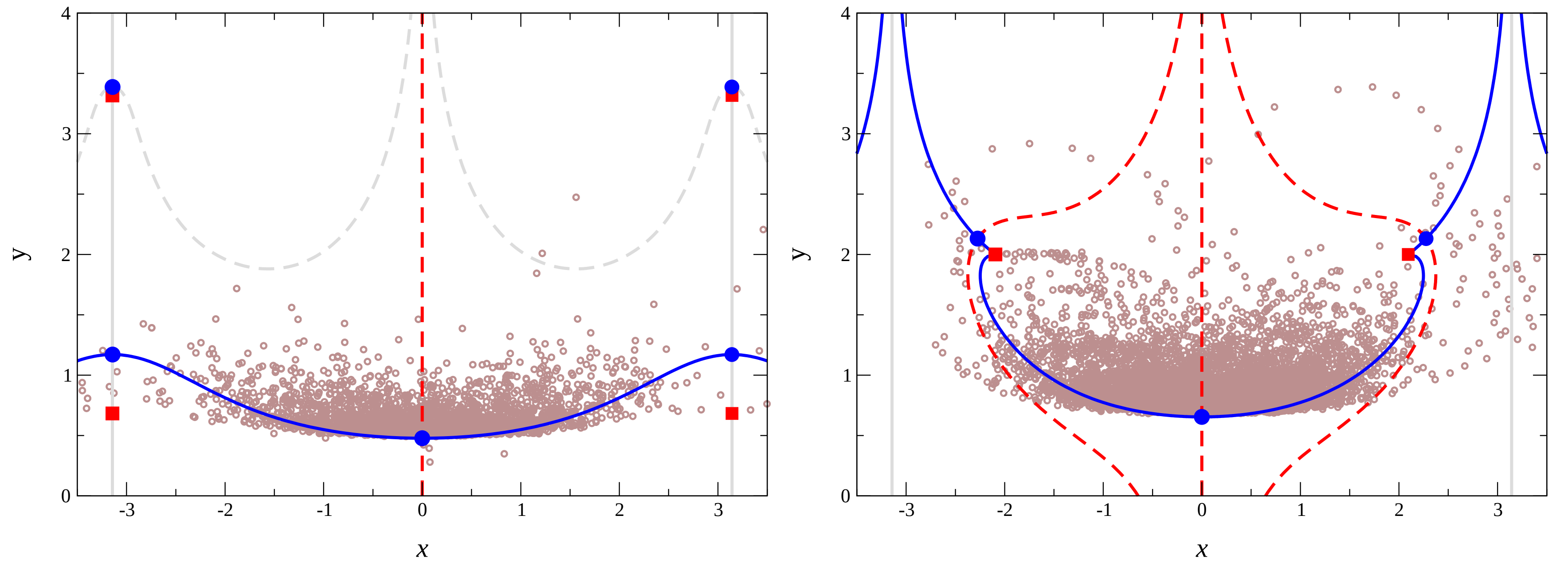


$\kappa = 1/2 < 1$ $\kappa = 2 > 1$

- arrows: Langevin drift
- blue dots: fixed points
- red squares: diverging drift, $\det = 0$

$\Rightarrow$ new feature: thimbles can end, $\mathrm{Im}S$ jumps

U(1) model with determinant



$\kappa = 1/2 < 1$ $\kappa = 2 > 1$

- dots: Langevin trajectory
- blue lines: contributing stable thimbles

Langevin distribution follows thimbles
spread in y direction when $\kappa > 1$

SU(2) model

special case $\beta = i$ Berges & Sexty 08

- *degenerate* critical point at $\cos z = i$, $\partial_z^2 S(z) = 0$
- thimbles can be computed analytically

$$v(u) = \frac{1}{\tan u}\left(u \pm \sqrt{u^2 - (1-u^2)\tan^2 u}\right)$$

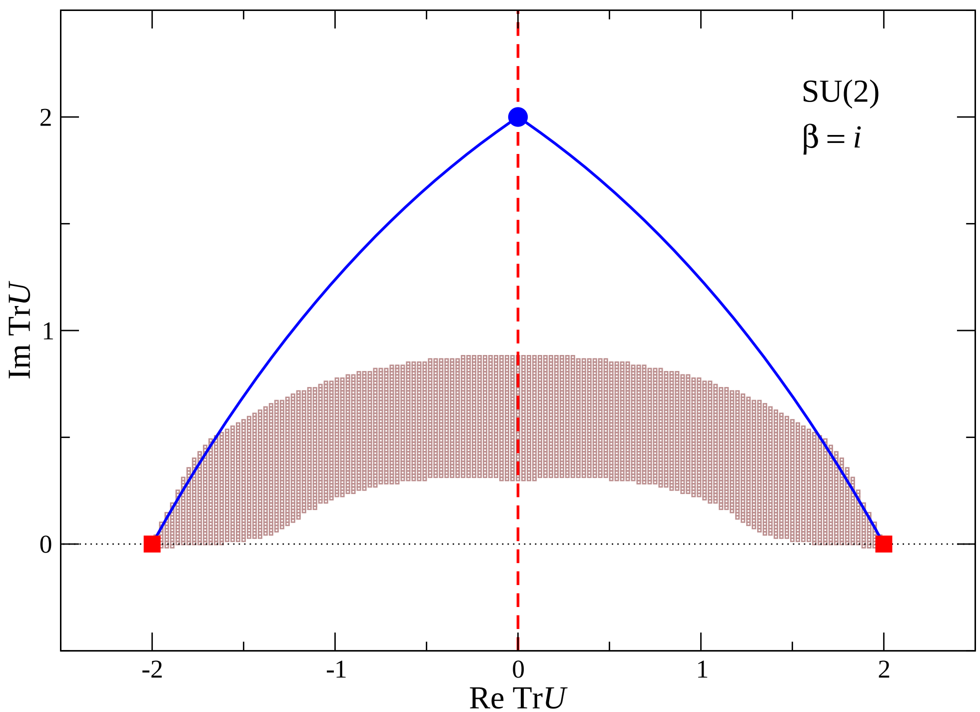
- in terms of

  $\frac{1}{2}\mathrm{Tr}\, U = \cos z$

  $= u + iv$

- CL distribution

  pinched by thimbles



Summary

exploring the complex plane: thimbles and Langevin

- location of distributions related but not identical
- weight distributions typically different
- repulsive fixed points in Langevin dynamics avoided

thimbles in simple models:

- *all* contributing thimbles should be included
- residual sign problem is relevant

in field theory both seem less stringent, why?