Complex actions, complex Langevin and Lefschetz thimbles

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Motivation

QCD partition function

at nonzero quark chemical potential

\[ [\det D(\mu)]^* = \det D(-\mu^*) \]

- fermion determinant is complex
- straightforward importance sampling not possible
- sign problem

\[ \Rightarrow \] QCD phase diagram has not yet been determined non-perturbatively
Outline

- complex actions
- Langevin dynamics
- thimble dynamics
- Langevin versus Lefschetz
- summary

GA, Lorenzo Bongiovanni, Erhard Seiler & Dénes Sexty, 1407.2090

Complex actions

one degree of freedom: \[ Z = \int dx \, e^{-S(x)} \]

complex holomorphic action \( S(z) \in \mathbb{C} \)

- numerical sign problem
- dominant configurations in the (path) integral?

- real and positive distribution \( P(x, y) \)?
Complex actions

various approaches relying on holomorphy:
go into the complex plane

- saddle point/steepest descent: Lefschetz thimbles
  Cristoforetti, Di Renzo, Mukherjee, Scorzato, (Schmidt) 12-14
  Fujii, Honda, Kato, Kikukawa, Komatsu, Sano 13
  Dunne, Unsal et al 12-14
  ...  

- complex Langevin dynamics/stochastic quantisation
  GA, Seiler, Sexty, Stamatescu,
  James, Bongiovanni, Giudice, Jaeger, Attanasio
  ...

see talk by Dénes Sexty for progress in gauge theories
Complex Langevin dynamics

Langevin dynamics:

- \dot{z} = -\partial_z S(z) + \eta \quad z = x + iy

associated Fokker-Planck equation (FPE)

\[ \dot{P}(x, y; t) = \left[ \partial_x (\partial_x + \text{Re}\partial_z S(z)) + \partial_y \text{Im}\partial_z S(z) \right] P(x, y; t) \]

(equilibrium) distribution in complex plane: \( P(x, y) \)

observables

\[ \langle O(x + iy) \rangle = \frac{\int dx dy \, P(x, y) O(x + iy)}{\int dx dy \, P(x, y)} \]

\( P(x, y) \) real and non-negative: no sign problem

criteria for correctness
Lefschetz thimbles

generalised saddle point integration/steepest descent:

extend definition of path integral

- Chern-Simons theories
- mathematical foundation in Morse theory

formulation:

- find \textit{all} stationary points $z_k$ of holomorphic action $S(z)$
- paths of steepest descent: stable thimbles $\mathcal{J}_k$
- paths of steepest ascent: unstable thimbles $\mathcal{K}_k$
- $\text{Im } S(z)$ constant along thimble $k$

integrate over stable thimbles, with proper weighting
Lefschetz thimbles

generalised saddle point integration/steepest descent:

- integrate over stable thimbles

\[
Z = \sum_{k} \left. m_k e^{-i \text{Im} S(z_k)} \int_{J_k} dz e^{-\text{Re} S(z)} \right.
\]

\[
= \sum_{k} m_k e^{-i \text{Im} S(z_k)} \int ds \ z'(s) e^{-\text{Re} S(z(s))}
\]

- intersection numbers: \( m_k = \langle C, K_k \rangle \)
  \( C = \) original contour, \( K_k = \) unstable thimble

- residual sign problem: complex Jacobian \( J(s) = z'(s) \)

- global sign problem: phases \( e^{-i \text{Im} S(z_k)} \)
Lefschetz thimbles

numerical Lefschetz approach

- find all saddle points/thimbles in field theory?
- integrate over dominant thimble $\mathcal{J}_0$ only

$$Z = e^{-i\text{Im}S(z_0)} \int_{\mathcal{J}_0} dz \ e^{-\text{Re}S(z)}$$

- motivated by universality
- no global sign problem
- residual sign problem remaining

validity?

- successful e.g. in interacting 4-dim Bose gas with $\mu \neq 0$
Langevin versus Lefschetz

two approaches in the complex plane:

- **Langevin**

\[
\langle O(z) \rangle = \frac{\int dx dy \ P(x, y) O(x + iy)}{\int dx dy \ P(x, y)}
\]

- **Lefschetz**

\[
\langle O(z) \rangle = \sum_k m_k e^{-i \text{Im} S(z_k)} \int_{\mathcal{J}_k} dz \ e^{-\text{Re} S(z)} O(z) \\
\sum_k m_k e^{-i \text{Im} S(z_k)} \int_{\mathcal{J}_k} dz \ e^{-\text{Re} S(z)}
\]

- two- versus one-dimensional
- real versus residual/global phases

relation? validity? \[\Rightarrow\] simple models
Langevin versus Lefschetz
Quartic model

\[ Z = \int_{-\infty}^{\infty} dx \ e^{-S} \quad S(x) = \frac{\sigma}{2} x^2 + \frac{\lambda}{4} x^4 \]

complex mass parameter \( \sigma = A + iB, \lambda \in \mathbb{R} \)

often used toy model \ Ambjorn & Yang 85, Klauder & Petersen 85, Okamoto et al 89, Duncan & Niedermaier 12

essentially analytical proof for CL*:

- CL gives correct result for all observables \( \langle x^n \rangle \)
- provided that \( A > 0 \) and \( A^2 > B^2 / 3 \)
- based on properties of the distribution \( P(x, y) \)
- follows from classical flow or directly from FPE

* GA, Seiler, Stamatescu 09 + James 11
Quartic model

- numerical solution of FPE for $P(x, y)$
- distribution is localised in a strip around real axis
- $P(x, y) = 0$ when $|y| > y_-$ with $y_- = 0.303$ for $\sigma = 1 + i$
Langevin versus Lefschetz

Lefschetz thimbles for quartic model

- critical points:
  
  \[ z_0 = 0 \]
  \[ z_{\pm} = \pm i \sqrt{\frac{\sigma}{\lambda}} \]

- thimbles can be computed analytically

  \[ \text{Im} S(z_0) = 0 \]
  \[ \text{Im} S(z_{\pm}) = -\frac{AB}{2\lambda} \]

- for \( A > 0 \): only 1 thimble contributes

  - integrating along thimble gives correct result, with inclusion of complex Jacobian
Quartic model: thimbles

compare thimble and FP distribution $P(x, y)$

- thimble and $P(x, y)$ follow each other
Quartic model: thimbles

compare thimble and FP distribution $P(x, y)$

- thimble and $P(x, y)$ follow each other
- however, weight distribution quite different

intriguing result: complex Langevin process finds the thimble – is this generic?
Langevin versus Lefschetz

compare evolution equations in more detail

- complex Langevin (CL) dynamics

\[
\dot{x} = - \text{Re} \partial_z S(z) + \eta \\
\dot{y} = - \text{Im} \partial_z S(z)
\]

- Lefshetz thimble dynamics, with \( z(t \to \infty) = z_k \)

\[
\dot{x} = - \text{Re} \partial_z S(z) \\
\dot{y} = + \text{Im} \partial_z S(z)
\]

\Rightarrow \text{ change in sign for } y \text{ drift}

Langevin:
- stable and unstable fixed points
- unstable runaways as \( y \to \pm \infty \)

thimbles:
- saddle points
- stable thimbles coming from \( y \to \pm \infty \)
Langevin versus Lefschetz

deform quartic model with linear term, break symmetry

\[ S(z) = \frac{\sigma}{2} z^2 + \frac{1}{4} z^4 + h z \]

\[ h \in \mathbb{C} \]

Langevin flow for \( \sigma = 1, h = 1 + i \)

- one stable/two unstable fixed points for CL
- \( y \rightarrow -\infty \) classical runaway trajectory
- two contributing thimbles (global phase problem) due to Stokes’ phenomenon (1847)
Langevin versus Lefschetz

histogram of $P(x, y)$ collected during CL simulation
Langevin versus Lefschetz

comparison of Langevin distribution with thimbles

- thimbles: both saddle points contribute
- CL: unstable fixed point avoided
- no role for second thimble in Langevin
⇒ distributions manifestly different
Other (more relevant) models

U(1) model with determinant

\[ Z = \int_{-\pi}^{\pi} dx \, e^{\beta \cos x} [1 + \kappa \cos(x - i\mu)] \]

- presence of \( \log \det \) of interest for CL and thimbles
  GA & Stamatescu 08, Mollgaard & Splittorff 13, Greensite 14

SU(2) one-link model with complex \( \beta \)

\[ Z = \int dU \, e^{-S(U)} \quad S(U) = -\frac{\beta}{2} \text{Tr} \, U \]

- solvable with CL in different ways (gauge fixing, gauge cooling, ...)

SEWM14, July 2014 – p. 14
U(1) model with determinant

\[ \kappa = 1/2 < 1 \quad \text{and} \quad \kappa = 2 > 1 \]

- arrows: Langevin drift
- blue dots: fixed points
- red squares: diverging drift, \( \det = 0 \)

\( \Rightarrow \) new feature: thimbles can end, \( \text{Im} S \) jumps
U(1) model with determinant

\[
\kappa = 1/2 < 1
\]

\[
\kappa = 2 > 1
\]

dots: Langevin trajectory
blue lines: contributing stable thimbles

Langevin distribution follows thimbles spread in \( y \) direction when \( \kappa > 1 \)
SU(2) model

special case $\beta = i$

- *degenerate* critical point at $\cos z = i$, $\partial_z^2 S(z) = 0$
- thimbles can be computed analytically

$$v(u) = \frac{1}{\tan u} \left( u \pm \sqrt{u^2 - (1 - u^2) \tan^2 u} \right)$$

- in terms of

$$\frac{1}{2} \text{Tr } U = \cos z$$
$$= u + iv$$

- CL distribution
  pinched by thimbles
Summary

exploring the complex plane: thimbles and Langevin

- location of distributions related but not identical
- weight distributions typically different
- repulsive fixed points in Langevin dynamics avoided

thimbles in simple models:

- all contributing thimbles should be included
- residual sign problem is relevant

in field theory both seem less stringent, why?