Fluctuation induced first order transition in the U(n)×U(n) models using chiral invariant expansion of the FRG flows

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Programs for Junior Scientists

1. Introduction

- The $U_L(n) \times U_R(n)$ symmetric matrix model is regarded as a low-energy effective model of QCD for $n = 2, 3$ [1]
- ϵ-expansion shows that there is no infrared stable fixed point of the β-functions if $n \geq 2$ [2]
- No stable IR fixed point: indirect evidence of a fluctuation induced first order transition
- Is it really one? Direct evidence only available for $n = 2$ [3,4]
- Goals:
  $\longrightarrow$ search for direct evidence for the order of the transition
  $\longrightarrow$ map the properties of the transition in the parameter space
  $\longrightarrow$ develop an approximation scheme suitable for phenomenological applications

2. Basics

- Model: dynamics of matrix field $M \in$ Lie$[U(n)]$ with two quartic interactions:

$$\mathcal{L} = \partial_\mu M \partial^\mu M^\dagger - m^2 \,\mathrm{Tr}\,(MM^\dagger) - \frac{g_1}{n^2}[\mathrm{Tr}\,(MM^\dagger)]^2 - \frac{g_2}{n}\,\mathrm{Tr}\,(MM^\dagger MM^\dagger)$$

- Symmetries:

$$M \longrightarrow U_R M U_L^\dagger,$$

which is equivalent to $U_V(n) \times U_A(n)$ as

$$M \longrightarrow VMV^\dagger, \qquad M \longrightarrow A^\dagger M A^\dagger$$

(with parameters $\theta^a_{V,A} = (\theta^a_R \pm \theta^a_L)/2$).

- Stability $\rightarrow$ constraints on param.'s: $g_1 + g_2 > 0$, $g_1 + ng_2 > 0$
  $\rightarrow$ Case I.: $\quad g_2 > 0, \quad g_1 + g_2 > 0$
  $\rightarrow$ Case II.: $\quad g_2 < 0, \quad g_1 + ng_2 > 0$
- Symmetry breaking pattern: [5]
  $\rightarrow$ Case I.: $\quad U(n) \times U(n) \longrightarrow U(n)$
  $\rightarrow$ Case II.: $\quad U(n) \times U(n) \longrightarrow U(n-1) \times U(n-1)$
  $\Longrightarrow$ only case I is compatible with QCD

3. Functional renormalization group

- Scale (k) dependent effective action (Γ_k):

$$\Gamma_k[\bar\phi] = W_k[J] - \int J\bar\phi - \frac{1}{2}\int \bar\phi R_k \bar\phi$$

$\longrightarrow$ R_k (regulator) suppresses modes with momenta $q \lesssim k$
$\longrightarrow$ $k = 0$: quantum effective action, $k = \Lambda$: classical action
$\longrightarrow$ satisfies a flow equation [6]

$$\partial_k \Gamma_k = \frac{1}{2} STr\left[\frac{1}{\Gamma_k^{(2)} + R_k}\partial_k R_k\right]$$

- Local potential approximation (LPA):

$$\Gamma_k[\bar\phi] \approx \int_x \Big(\partial_\mu \bar\phi(x)\partial^\mu \bar\phi(x) - V_k(x)\Big)$$

- Litim's 3D regulator:

$$R_k(q) = (k^2 - \vec{q}^{\,2})\Theta(k^2 - \vec{q}^{\,2})$$

- Flow of the local potential at temperature T:

$$\partial_k V_k = \frac{k^4}{6\pi^2} T \sum_{\omega_n}\sum_i \frac{1}{\omega_n^2 + k^2 + M_i^2}$$

4. Symmetry requirements

- The local potential depends on chiral invariant tensors $\{I_i\}$ $(i = 1, \dots n)$

$$V_k = V_k(I_1, I_2, \dots I_n)$$

- Set of choice:

$$I_1 = Tr[M^\dagger M],$$
$$I_2 = Tr\left[M^\dagger M - \frac{1}{n}Tr[M^\dagger M]\right]^2$$
$$\dots$$
$$I_n = Tr\left[M^\dagger M - \frac{1}{n}Tr[M^\dagger M]\right]^n$$

- The mass matrix entering to the r.h.s of the flow eq. can be obtained via Leibnitz's rule:

$$M^2_{ab} = \frac{\partial^2 V_k}{\partial I_i I_j}\frac{\partial I_i}{\partial \phi_a}\frac{\partial I_j}{\partial \phi_b} + \frac{\partial V_k}{\partial I_i}\frac{\partial^2 I_i}{\partial \phi_a \partial \phi_b}$$

- Expected symmetry breaking pattern: $M \sim \mathbf{1}$
  $\longrightarrow \quad I_1 = v_0^2/2, \quad I_{n>1} = 0$

5. Chiral invariant expansion

- Based on the expected $M \sim \mathbf{1}$ symmetry breaking pattern
  $\longrightarrow$ V_k is expanded around this configuration

$$V_k(I_1, I_2, \dots I_n) = U_k(I_1) + \sum_{\{\alpha\}} C_k^{(\alpha)}(I_1)\prod_{i=2}^n I_i^{\alpha_i}$$

- The flow eq. of V_k determines the evolution of U_k and all $C_k^{(\alpha)}$
- V_k on an n-dim. grid $\longleftrightarrow$ coefficients on a 1-dim. grid
- Three steps of obtaining the flows of the coefficients:
  $\longrightarrow$ calculate the mass matrices using the most general (diagonal) background $M = v_a T^a$
  $\longrightarrow$ expand the r.h.s. of the flow equation around the $M \sim \mathbf{1}$ symmetry breaking pattern
  $\longrightarrow$ identify the invariants and obtain the respective flow equations of the coefficients

6. Approximate solution

- The chiral invariant expansion is approximated as

$$V_k \approx U_k(I_1) + C_k(I_1)\cdot I_2$$

- Consequences:
  $\longrightarrow$ 2 component condensate is sufficient
  $\longrightarrow$ identification of the invariants is easy
  $\longrightarrow$ two coupled flow equations for U_k and C_k
- The excitations appear: σ, a_0, π

$$\partial_k U_k(I_1) = \frac{k^4 T}{6\pi^2}\sum_{\omega_n}\left[\frac{n^2}{\omega_n^2 + E_\pi^2} + \frac{n^2-1}{\omega_n^2 + E_{a_0}^2} + \frac{1}{\omega_n^2 + E_\sigma^2}\right]$$

$$\partial_k C_k(I_1) = \frac{k^4 T}{6\pi^2}\sum_{\omega_n} F(I_1;\omega_n)$$

with

$$\begin{aligned}
F = {} & \frac{4(3C_k + 2I_1 C_k')^2/n}{(\omega_n^2 + E_{a_0}^2)^2(\omega_n^2 + E_\sigma^2)} + \frac{128 C_k^5 I_1^3/n}{(\omega_n^2 + E_\pi^2)^3(\omega_n^2 + E_{a_0}^2)^3} \\
& + \frac{4C_k\left(4C_k(n^2-3) + (1-4n^2)I_1 C_k'\right)/n}{(\omega_n^2 + E_{a_0}^2)^3} \\
& + \frac{4\left(3C_k C_k' I_1 + 4I_1^2 C_k' + C_k(3C_k - 2C_k'' I_1^2)\right)/n}{(\omega_n^2 + E_{a_0}^2)(\omega_n^2 + E_\sigma^2)^2} \\
& + \frac{64 C_k^3 I_1^2 (C_k - I_1 C_k')/n}{(\omega_n^2 + E_\pi^2)^2(\omega_n^2 + E_{a_0}^2)^3} - \frac{48 C_k^2 I_1^2 C_k'}{(\omega_n^2 + E_\pi^2)(\omega_n^2 + E_{a_0}^2)^3} \\
& + \frac{6C_k + (1-2n^2)I_1 C_k'}{(\omega_n^2 + E_{a_0}^2)^2}\frac{1}{I_1} - \frac{6C_k + 9I_1 C_k' + 2I_1^2 C_k''}{(\omega_n^2 + E_\sigma^2)^2 I_1} \\
& + \frac{4C_k(6C_k + 9I_1 C_k' + 2I_1^2 C_k'')/n}{(\omega_n^2 + E_{a_0}^2)(\omega_n^2 + E_\sigma^2)^2} \\
& - \frac{2C_k(12C_k + 2(1-2n^2)I_1 C_k')/n}{(\omega_n^2 + E_{a_0}^2)^3}
\end{aligned}$$

7. β-functions

- In the dimensionally reduced theory, the flow equations give account of the β-functions
- Dimensional reduction: formally $T \to \infty$
- Assumption: V_k has the form of the classical action

$$U_k(I_1) = m_k^2 I_1 + \frac{4\pi^2}{3}\left(g_{1,k} + \frac{g_{2,k}}{n}\right)I_1^2$$

$$C_k(I_1) = \frac{4\pi^2}{3}g_{2,k}$$

- β-functions in $d = 4 - \epsilon$ dim.: ($\bar g_{i,k}$: dimensionless couplings)

$$\beta_1 = k\frac{\partial \bar g_{1,k}}{\partial k} = -\epsilon \bar g_{1,k} + \frac{n^2+4}{3}\bar g_{1,k}^2 + \frac{4n}{3}\bar g_{1,k}\bar g_{2,k} + \bar g_{2,k}^2$$

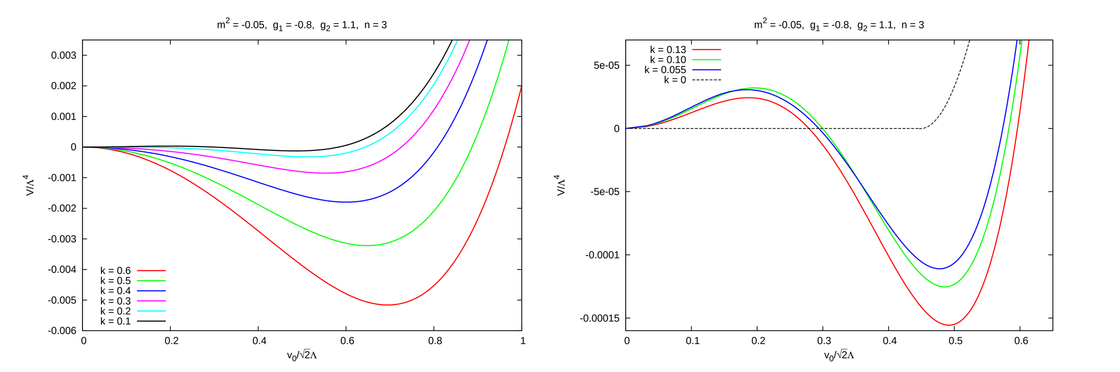
$$\beta_2 = k\frac{\partial \bar g_{2,k}}{\partial k} = -\epsilon \bar g_{2,k} + \frac{2n}{3}\bar g_{2,k}^2 + 2\bar g_{1,k}\bar g_{2,k}$$

$\longrightarrow$ these are exactly the same results as of the ϵ-expansion [2]

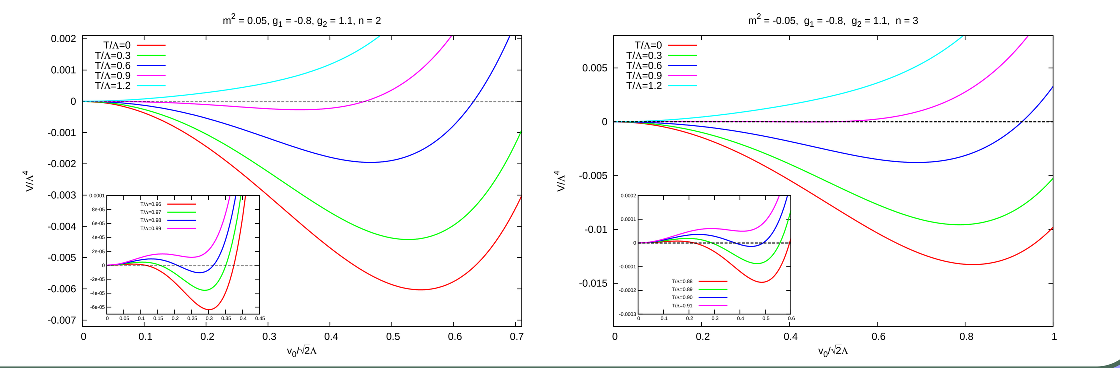
8. Numerics

- Grid method:
  $\longrightarrow$ functions stored typically $I_1/\Lambda^2 \in [0, 2]$
  $\longrightarrow$ step size on the grid: 10^{-3}
  $\longrightarrow$ field derivatives: calculated using the 7-point formula
  $\longrightarrow$ solution: adaptive Runge-Kutta method
  $\longrightarrow$ typical step size in k-space: $10^{-5}\Lambda$
- Reaching $k \to 0$ is very costly numerically
  $\longrightarrow$ $T_c(k = 0)$ and the jump of v_0 at $k = 0$ at the transition point are obtained by extrapolation
  $\longrightarrow$ in both cases $f(k) = a + b \cdot k^c$ is fitted

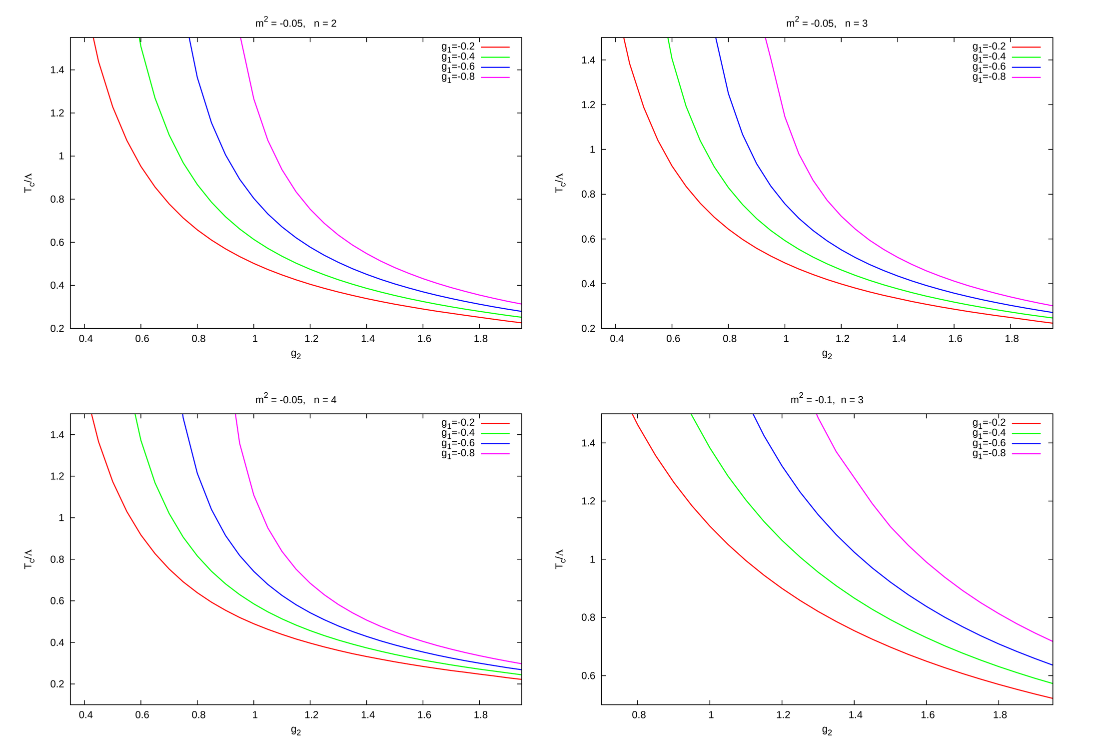
9. Properties of the flow



- V_k is becoming convex as $k \to 0$
- For intermediate scales V_k is not convex
  $\longrightarrow$ T_c can only be defined as $T_c = \lim_{k\to 0} T_c(k)$
  $\longrightarrow$ $k = 0$ is difficult numerically $\Rightarrow$ extrapolation
- It could make sense to stop the flow at $\hat k = 1/L (= 0.2$ on fig.)
- Transition is always of first order

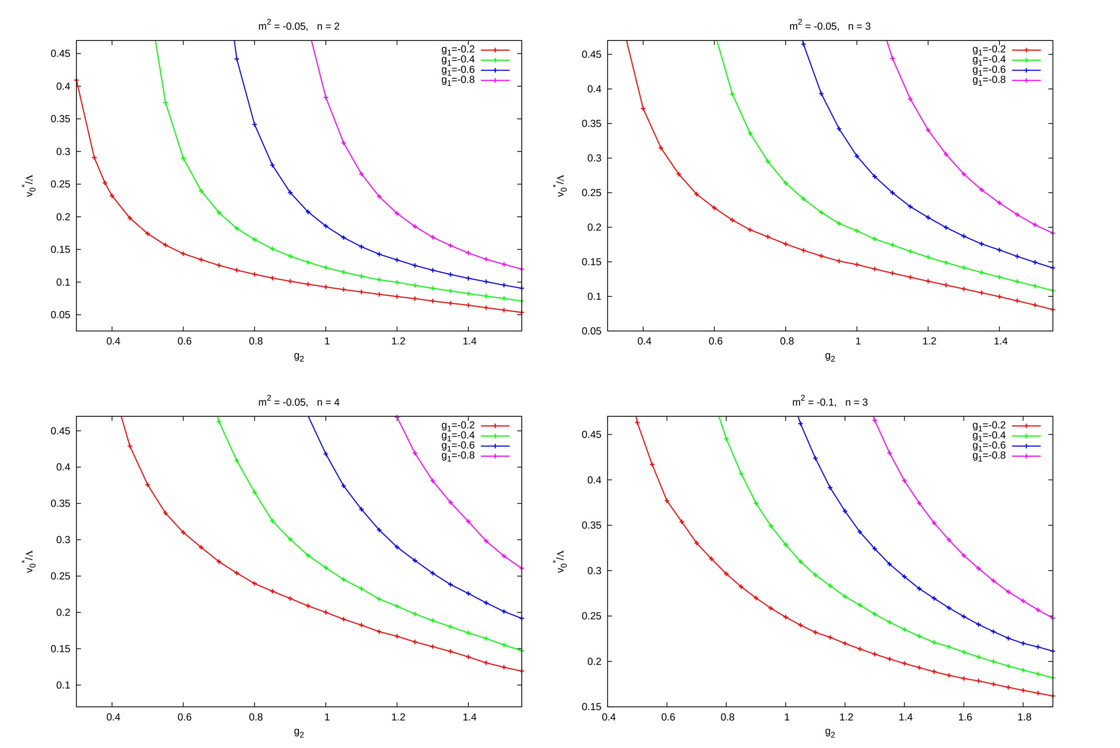


10. Critical Temperature (T_c)



- Smaller g_1 leads to smaller T_c as a function of g_2
- T_c is not very sensitive to the flavor number n
- T_c grows with increasing $|m^2|$

11. Jump of the order parameter (v_0^*)



- v_0^* is more sensitive to the flavor number n than T_c
- Increasing $|m^2|$ leads to a bigger jump of the condensate
- Both T_c and v_0^* diverges as $g_2 \to |g_1|$

12. Conclusions

- $U(n) \times U(n)$ model: low energy effective model of QCD
- Obtaining the effective potential using FRG formalism
  $\longrightarrow$ Litim's 3D regulator + Local Potential Approximation
  $\longrightarrow$ Finite temperature treatment
- Effective potential: represented by a chiral invariant expansion
- Direct evidence of a first order transition for arbitrary n
- Possible extensions with finite quark masses and anomaly

13. References

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