An exact solution of the Schwinger-Dyson equation in de Sitter space time

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SEWM 2014, EPFL Lausanne, July 17th, 2014

Based on
In preparation , F.G, Julien Serreau
Why Quantum Field Theory in de Sitter space?

- First step towards Quantum Gravity
- Maximally symmetric space-time
- Perfect test lab for periods of accelerated expansion of the Universe

**Specific effects**
Outline

1. Momentum representation of correlators

2. Resummation of non-local logarithms

3. Solution and discussion

4. Conclusions and prospects
Radiative corrections

- Scalar field $\varphi$ with quartic self-interaction $S_{\text{int}} = -\int d^4x \sqrt{-g} \frac{\lambda}{4!}\varphi^4$

- Infrared divergent radiative corrections

$$\sim \frac{\lambda H^4}{m^2}$$

$$\sim \frac{\lambda^2 H^6}{m^2} \ln p \sim t$$

- Line element in conformal time $-\infty < \eta < 0$ and comoving spatial coordinates $X$

$$ds^2 = \eta^{-2}(-d\eta^2 + dX.dX)$$

$\rightarrow$ Intrinsically out of equilibrium.

1/$N$ techniques, Non-Equilibrium Renormalisation
Group, Stochastic approaches, 2-PI techniques, etc...

- With the Schwinger-Dyson equation $G^{-1}(x, x') = G_0^{-1}(x, x') - \Sigma(x, x')$
Momentum representation of correlators

R. Parentani, J. Serreau ('12)

- Searching for de Sitter invariant solutions. In principle
  \[ G(x, x') = G(z) \]
- But Inverting \( G^{-1}(z) = G_0^{-1}(z) - \Sigma(z) \) is "hard".
- In Minkowsky: momentum representation \( \rightarrow \) algebraic equation "easy"
  
  Not (fully) possible in de Sitter space-time

- Exploit partly de Sitter space symmetries

  \[ \rightarrow \text{Scaling property}: \ G(\eta, \eta', K) \equiv \frac{1}{K} \hat{G}(-K\eta, -K\eta') = \frac{1}{K} \hat{G}(p, p') \]
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  Physical momentum representation of correlators
Scaling property consistent with the Schwinger-Dyson equation, i.e,

\[ \hat{G}^{-1}(p, p') = \hat{G}_0^{-1}(p, p') - \hat{\Sigma}(p, p') \]

trade contour:

Momentum flow like SD-equations

\[
\begin{align*}
\left[ \partial_p^2 + 1 - \frac{\nu^2 - \frac{1}{4}}{p^2} \right] \hat{F}(p, p') &= \int_{p'}^{+\infty} ds \hat{\Sigma}_F(p, s)\hat{\rho}(s, p') \\
&\quad - \int_{p}^{+\infty} ds \hat{\Sigma}_\rho(p, s)\hat{F}(s, p') \\
\left[ \partial_p^2 + 1 - \frac{\nu^2 - \frac{1}{4}}{p^2} \right] \hat{\rho}(p, p') &= - \int_{p}^{p'} ds \hat{\Sigma}_\rho(p, s)\hat{\rho}(s, p')
\end{align*}
\]
Momentum representation of correlators

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- Scaling property consistent with the Schwinger-Dyson equation, i.e,

\[ \hat{G}^{-1}(p, p') = \hat{G}_0^{-1}(p, p') - \hat{\Sigma}(p, p') \]

- Trade contour:

\[
\begin{align*}
\partial_p^2 + 1 - \frac{\nu^2 - \frac{1}{4}}{p^2} & \quad \hat{F}(p, p') = \int_{p'}^{+\infty} ds \ \hat{\Sigma}_F(p, s)\hat{\rho}(s, p') \\
& - \int_p^{+\infty} ds \ \hat{\Sigma}_\rho(p, s)\hat{F}(s, p') \\
\partial_p^2 + 1 & \quad \hat{\rho}(p, p') = -\int_{p'}^{p} ds \ \hat{\Sigma}_\rho(p, s)\hat{\rho}(s, p')
\end{align*}
\]

(0 + 1) effective description

- Momentum flow like SD-equations
Resummation of non-local logarithms

The resummation

- Remember two type of divergences

\[ \sim \frac{\lambda H^4}{m^2} \quad \text{\rightarrow Mass term } M \]

\[ \sim \frac{\lambda^2 H^6}{m^2} \ln p \sim t \quad \text{\rightarrow This talk} \]

- Take for granted the mass generation mechanism

[A.A.Starobinsky, J.Yokoyama ('96)], [B.Garbrecht, G.Rigopoulos ('11)], [Serreau ('11)], [D.Boyanosky ('12)],...

- We wish to calculate

\[ + + + \]

- Schwinger-Dyson equation

\[ \hat{G}^{-1} = \hat{G}_M^{-1} - \hat{\Sigma}_{nl}[\hat{G}_M] \quad \text{with} \quad \hat{\Sigma}_{nl} = \]

\[ \begin{array}{c}
\hat{G}^{-1} = \hat{G}_M^{-1} - \hat{\Sigma}_{nl}[\hat{G}_M] \\
\text{with} \\
\hat{\Sigma}_{nl} = \end{array} \]
**Infrared self-energy**

\[ \sum_{nl} = \begin{array}{c} \text{for super-horizon modes } p, p' \lesssim H \end{array} \]

- de Sitter "plane waves": \( \hat{G}_M(p, p') = \frac{\pi}{4} \sqrt{pp'} H_\nu(p) H_\nu^*(p') \)

- Infrared enhancement of correlators \( \rightarrow \) Infrared contributions dominate
  
  \[ \hat{F}^{\text{IR}}_M(p, p') = \sqrt{pp'} F_\nu e^{-\nu(x+x')} , \]
  
  \[ \hat{\rho}^{\text{IR}}_M(p, p') = -\sqrt{pp'} \mathcal{P}_\nu(x - x'), \]

  with \( x = \ln\left(\frac{p}{H}\right) \) and \( \mathcal{P}_\nu(x) = \sinh(\nu x) / \nu \)

- The self energies are
  
  \[ \hat{\Sigma}^{\text{IR}}_F(p, p') = -(pp')^{-3/2} F_\nu \sigma \rho s(x) s(x') \]
  
  \[ \hat{\Sigma}^{\text{IR}}_\rho(p, p') = (pp')^{-3/2} \sigma \rho \sigma (x - x') \]

  with \( s(x) = e^{-(\nu - 2\varepsilon)x} \), \( \sigma(x) = \mathcal{P}_\nu(x)e^{-2\varepsilon|x|} \), \( \nu \equiv \frac{d}{2} - \varepsilon \), i.e., \( \varepsilon \simeq \frac{M^2}{d} \) and

  \[ \sigma \rho \sim \frac{\chi^2}{M^4} \]
Infrared self-energy

\[ \hat{\Sigma}_{nl} = \text{for super-horizon modes } p, p' \lesssim H \]

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- Infrared enhancement of correlators \( \rightarrow \) Infrared contributions dominate
  \[ \hat{F}^\text{IR}_M(p, p') = \sqrt{pp'} F_\nu e^{-\nu(x+x')}, \]
  \[ \hat{\rho}^\text{IR}_M(p, p') = -\sqrt{pp'} \mathcal{P}_\nu(x - x'), \]

**Factorisation property**

- The self energies are
  \[ \hat{\Sigma}^\text{IR}_F(p, p') = -(pp')^{-3/2} F_\nu \sigma_\rho s(x)s(x') \]
  \[ \hat{\Sigma}^\text{IR}_\rho(p, p') = (pp')^{-3/2} \sigma_\rho \sigma(x - x') \]
  with \( s(x) = e^{-(\nu-2\varepsilon)x} \), \( \sigma(x) = \mathcal{P}_\nu(x) e^{-2\varepsilon|x|} \), \( \nu \equiv \frac{d}{2} - \varepsilon \), i.e., \( \varepsilon \simeq \frac{M^2}{d} \) and
  \[ \sigma_\rho \sim \frac{\chi^2}{M^4} \]
One dimensional dynamics

- One dimensional reduction: \[ \hat{\rho}(p, p') = \sqrt{pp'} \rho(x - x') \]

- Statistical part of the formal solution:
  \[ \hat{F}^{\text{IR}}(p, p') = \sqrt{pp'} \tilde{F}_\nu \left\{ f_R(x) f_R(x') + \sigma_\rho f_\sigma(x) f_\sigma(x') \right\} \]

  with \( f_R(x) = \rho'(x) - \nu \rho(x) \) and \( f_\sigma(x) = \int_0^x dy \rho(x - y) s(y) \)

- Non trivial part of the spectral function:
  \[ \rho''(x) - \nu^2 \rho(x) = \sigma_\rho \int_0^x dy \sigma(x - y) \rho(y) \]

This is the equation resuming all
Full solution

- Exact solution
- Non-trivial spectral part:

\[
\rho(x) = \frac{1}{2\tilde{\nu}} \left\{ (\tilde{\nu} + \varepsilon)\mathcal{P}_{\tilde{\nu}+}(x) + (\tilde{\nu} - \varepsilon)\mathcal{P}_{\tilde{\nu}-}(x) + \text{sign}(x) \left[ \mathcal{P}'_{\tilde{\nu}+}(x) - \mathcal{P}'_{\tilde{\nu}-}(x) \right] \right\} e^{-\varepsilon|x|}
\]

- Statistical parts:

\[
f_R(x) = \left\{ (\nu_- + \tilde{\nu}_+)A_{\tilde{\nu}}(\tilde{\nu}_+)e^{-\tilde{\nu}_+x} + (\nu_- - \tilde{\nu}_+)A_{\tilde{\nu}}(-\tilde{\nu}_+)e^{\tilde{\nu}_+x} \right\} e^{\varepsilon x} + (\tilde{\nu} \rightarrow -\tilde{\nu})
\]

and

\[
f_\sigma(x) = \left\{ \frac{A_{\tilde{\nu}}(\tilde{\nu}_+)}{\nu_- - \tilde{\nu}_+}e^{-\tilde{\nu}_+x} + \frac{A_{\tilde{\nu}}(-\tilde{\nu}_+)}{\nu_- + \tilde{\nu}_+}e^{\tilde{\nu}_+x} \right\} e^{\varepsilon x} + (\tilde{\nu} \rightarrow -\tilde{\nu}),
\]

with

\[
A_{\tilde{\nu}}(z) = (z + \tilde{\nu} + \varepsilon)/4\tilde{\nu}z, \quad \tilde{\nu} = \sqrt{\nu^2 + \frac{\sigma_\rho}{4\varepsilon^2}}, \quad \varepsilon \simeq \frac{M^2}{d}
\]

and

\[
\tilde{\nu}_\pm^2 = \nu^2 \pm 2\varepsilon\tilde{\nu} + \varepsilon^2
\]
Moderate infrared

- Simpler form for $|x| \gtrsim 1$ and at first order in infrared enhancement $1/M^2$
- Superposition of two massive solutions:
  \[
  \rho(x) = \left\{ c_+ \mathcal{P}_{\bar{\nu}_+}(x) + c_- \mathcal{P}_{\bar{\nu}_-}(x) \right\} e^{-\varepsilon|x|}
  \]
  with $c_\pm = (\bar{\nu} \pm \nu)/2\bar{\nu}$
- Similarly for the statistical part
  \[
  \hat{F}_\text{IR}(p,p') = \sqrt{pp'} F_{\nu} \left\{ c_+ e^{-\bar{\nu}_+(x+x')} + c_- e^{-\bar{\nu}_-(x+x')} \right\} e^{\varepsilon(x+x')}
  \]
- Interpolate between two free massive solutions

\[
\begin{array}{c}
\text{ln } \mu/p \\
0 \quad 1 \quad 2 \quad 3 \quad 4
\end{array}
\]

\[
\begin{array}{c}
F(p,p)/F_M(p,p) \\
0.8 \quad 1 \quad 1.2 \quad 1.4
\end{array}
\]

- Agreement on "Dynamical mass":
  \[
  \langle \varphi^2(x) \rangle = \int \frac{d^d p}{(2\pi)^d} \frac{\hat{F}(p,p)}{p} \propto \frac{1}{m_{\text{dyn}}^2}
  \]
Conclusion

Conclusions

- Exact solution of the Schwinger-Dyson equation.
- Rich and infrared finite structure of the propagator.
- Dangerous logarithms are resummed in modified power laws (c.f. anomalous dimensions)

Prospects

- Non-perturbative methods: large-$N$ [in preparation], 2-PI, ...
- Extend the method to quasi de Sitter space-time: Quantum corrections to inflation.
  
  [[Weinberg ('05)], [M.van der Meulen, J.Smit ('07)], [Sloth ('07)], [M.Herranen, T.Markkanen, A.Tranberg ('13)]]