Continuum Results of the Heavy Quark Momentum Diffusion Coefficient $\kappa$

Olaf Kaczmarek

University of Bielefeld

in collaboration with

A.Francis, M. Laine, M.Müller, T.Neuhaus, H.Ohno


Strong and Electroweak Matter 2014
Lausanne
17.07.2014
Transport Coefficients are important ingredients into hydro/transport models for the evolution of the system. Usually determined by matching to experiment (see right plot)

Need to be determined from QCD using first principle lattice calculations!

here heavy flavour:

Heavy Quark Diffusion Constant $D$
[H.T.Ding, OK et al., PRD86(2012)014509]

Heavy Quark Momentum Diffusion $\kappa$

or for light quarks:

Light quark flavour diffusion

Electrical conductivity
[A.Francis, OK et al., PRD83(2011)034504]

Motivation - Transport Coefficients

- **Transport Coefficients**
- Hydro/transport models
- Evolution of the system
- Determined by matching to experiment
- Determined from QCD using first principle lattice calculations

Graphs and plots showing data and comparisons to theoretical predictions.
Transport coefficients from Lattice QCD – Flavour Diffusion

Transport coefficients usually calculated using correlation function of conserved currents

\[ G(\tau, p, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega, p, T) K(\tau, \omega, T) \]

\[ K(\tau, \omega, T) = \frac{\cosh \left( \omega (\tau - \frac{1}{2T}) \right)}{\sinh \left( \frac{\omega}{2T} \right)} \]

Lattice observables:

\[ G_{\mu\nu}(\tau, \vec{x}) = \langle J_\mu(\tau, \vec{x}) J^\dagger_\nu(0, \vec{0}) \rangle \]

\[ J_\mu(\tau, \vec{x}) = 2\kappa Z V \bar{\psi}(\tau, \vec{x}) \Gamma_\mu \psi(\tau, \vec{x}) \]

\[ G_{\mu\nu}(\tau, \vec{p}) = \sum_{\vec{x}} G_{\mu\nu}(\tau, \vec{x}) e^{i\vec{p}\cdot\vec{x}} \]

only correlation functions calculable on lattice but

Transport coefficient determined by slope of spectral function at \( \omega=0 \) (Kubo formula)

\[ D = \frac{\pi}{3\chi_{00}} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p} = 0, T)}{\omega T} \]
Different contributions and scales enter in the spectral function

- **continuum at large frequencies**
- **possible bound states at intermediate frequencies**
- **transport contributions at small frequencies**

notoriously difficult to extract from correlation functions

\[
G(\tau, \vec{p}, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho(\omega, \vec{p}, T) K(\tau, \omega, T)
\]

+ zero-mode contribution at \(\omega=0\):

\[
\rho(\omega) = 2\pi \chi_{00} \omega \delta(\omega)
\]

+ (narrow) transport peak at small \(\omega\):

\[
\rho(\omega \ll T) = 2\chi_{00} \frac{T}{M} \frac{\omega \eta}{\omega^2 + \eta^2}, \quad \eta = \frac{T}{MD}
\]
Heavy Quark Effective Theory (HQET) in the large quark mass limit

for a single quark in medium

leads to a (pure gluonic) “color-electric correlator”

\[ G_E(\tau) \equiv -\frac{1}{3} \sum_{i=1}^{3} \frac{\left\langle \text{Re Tr} \left[ U(\frac{1}{T}; \tau) gE_i(\tau, 0) U(\tau; 0) gE_i(0, 0) \right] \right\rangle}{\left\langle \text{Re Tr}[U(\frac{1}{T}; 0)] \right\rangle} \]

Heavy quark (momentum) diffusion:

\[ \kappa = \lim_{\omega \to 0} \frac{2T \rho_E(\omega)}{\omega} \quad D = \frac{2T^2}{\kappa} \]
can be related to the thermalization rate:

\[ \eta_D = \frac{\kappa}{2M_{kin}T} \left( 1 + O\left( \frac{\alpha_s^{3/2}T}{M_{kin}} \right) \right) \]


very poor convergence

\[ \rightarrow \text{Lattice QCD study required in the relevant temperature region} \]

in contrast to a narrow transport peak, from this a smooth limit

\[ \frac{\kappa}{T^3} = \lim_{\omega \to 0} \frac{2T \rho_E(\omega)}{\omega} \]

is expected

Qualitatively similar behaviour also found in AdS/CFT [S.Gubser, Nucl.Phys.B790 (2008)175]
due to the gluonic nature of the operator, signal is extremely noisy

→ **multilevel** combined with **link-integration** techniques to improve the signal


[Parisi, Petronzio, Rapuano PLB 128 (1983) 418, and de Forcrand PLB 151 (1985) 77]
heavy quark momentum diffusion constant – tree-level improvement

normalized by the LO-perturbative correlation function:

\[ G_{\text{norm}}(\tau T) \equiv \frac{G_{\text{cont}}^{\text{LO}}(\tau T)}{g^2 C_F} = \pi^2 T^4 \left[ \frac{\cos^2(\pi \tau T)}{\sin^4(\pi \tau T)} + \frac{1}{3 \sin^2(\pi \tau T)} \right] \]

and renormalized using NLO renormalization constants \( Z(g^2) \)
Heavy Quark Momentum Diffusion Constant – Tree-Level Improvement


lattice cut-off effects visible at small separations (left figure)

⇒ tree-level improvement (right figure) to reduce discretization effects

\[ G_{\text{cont}}^{\text{LO}}(\tau T) = G_{\text{lat}}^{\text{LO}}(\tau T) \]

leads to an effective reduction of cut-off effect for all \( \tau T \)
Quenched Lattice QCD on large and fine isotropic lattices at $T \approx 1.4 \, T_c$

- standard Wilson gauge action
- algorithmic improvements to enhance signal/noise ratio
- fixed aspect ratio $N_s/N_t = 4$, i.e. fixed physical volume $(2 \text{fm})^3$
- perform the continuum limit, $a \to 0 \leftrightarrow N_t \to \infty$
- determine $\kappa$ in the continuum using an Ansatz for the spectral fct. $\rho(\omega)$

<table>
<thead>
<tr>
<th>$N_\sigma$</th>
<th>$N_\tau$</th>
<th>$\beta$</th>
<th>$1/a [\text{GeV}]$</th>
<th>$a [\text{fm}]$</th>
<th>#Conf</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>64</td>
<td>16</td>
<td>6.872</td>
<td>7.16</td>
<td>0.03</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>96</td>
<td>24</td>
<td>7.192</td>
<td>10.4</td>
<td>0.019</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>144</td>
<td>36</td>
<td>7.544</td>
<td>15.5</td>
<td>0.013</td>
<td>362</td>
<td></td>
</tr>
<tr>
<td>192</td>
<td>48</td>
<td>7.793</td>
<td>20.4</td>
<td>0.010</td>
<td>223</td>
<td></td>
</tr>
</tbody>
</table>
Heavy Quark Momentum Diffusion Constant – Lattice results

finest lattices still quite noisy at large $\tau T$
but only small cut-off effects at intermediate $\tau T$
cut-off effects become visible at small $\tau T$
need to extrapolate to the continuum
perturbative behavior in the limit $\tau T \to 0$

allows to perform continuum extrapolation, $a \to 0 \iff N_t \to \infty$, at fixed $T = 1/a N_t$
Heavy Quark Momentum Diffusion Constant – Continuum extrapolation

well behaved continuum extrapolation for $0.05 \leq \tau T \leq 0.5$

finest lattice already close to the continuum

coarser lattices at larger $\tau T$ close to the continuum

how to extract the spectral function from the correlator?

finest lattices still quite noisy at large $\tau T$ but only small cut-off effects at intermediate $\tau T$

cut-off effects become visible at small $\tau T$

need to extrapolate to the continuum

perturbative behavior in the limit $\tau T \to 0$
Model spectral function: transport contribution + NLO [Y.Burnier et al. JHEP 1008 (2010) 094]

\[
\rho_{\text{model}}(\omega) \equiv \max\left\{ \rho_{\text{NLO}}(\omega), \frac{\omega \kappa}{2T} \right\}
\]

\[
G_{\text{model}}(\tau) \equiv \int_0^\infty \frac{d\omega}{\pi} \rho_{\text{model}}(\omega) \frac{\cosh\left(\frac{1}{2} - \tau T\right)}{\sinh\frac{\omega}{2T}} \frac{\omega}{T}
\]

some contribution at intermediate distance/frequency seems to be missing
Model spectral function: transport contribution + NLO + correction

\[ \rho_{\text{model}}(\omega) \equiv \max \left\{ A \rho_{\text{NLO}}(\omega) + B \omega^3, \frac{\omega \kappa}{2T} \right\} \]

\[ G_{\text{model}}(\tau) \equiv \int_0^\infty \frac{d\omega}{\pi} \rho_{\text{model}}(\omega) \frac{\cosh \left( \frac{1}{2} - \tau T \right) \frac{\omega}{T}}{\sinh \frac{\omega}{2T}} \]

Result of the fit to \( \rho_{\text{model}}(\omega) \) with three parameters: \( \kappa, A, B \)

\( \kappa/T^3 = 2.4(6) \)

\( \Rightarrow \) First continuum estimate of \( \kappa \):

(still preliminary)
Heavy Quark Momentum Diffusion Constant – Model Spectral Function

Model spectral function: transport contribution + NLO + correction

\[ \rho_{\text{model}}(\omega) \equiv \max \left\{ A \rho_{\text{NLO}}(\omega) + B \omega^3, \frac{\omega \kappa}{2T} \right\} \]

\[ G_{\text{model}}(\tau) \equiv \int_0^\infty \frac{d\omega}{\pi} \rho_{\text{model}}(\omega) \frac{\cosh \left( \frac{1}{2} - \tau T \right) \frac{\omega}{T}}{\sinh \frac{\omega}{2T}} \]

used to fit the continuum extrapolated data

\[ \kappa/T^3 = \lim_{\omega \to 0} \frac{2T \rho_E(\omega)}{\omega} \simeq 2.4(6) \]

result of the fit to \( \rho_{\text{model}}(\omega) \)

\[ A \rho_{\text{NLO}}(\omega) + B \omega^3 \]

NLO perturbation theory

\[ \frac{\omega \kappa}{2T} \text{ small but relevant contribution at } \tau T > 0.2! \]
Conclusions and Outlook

Continuum extrapolation for the color electric correlation function
extracted from Quenched Lattice QCD

\[ G_E(\tau) \equiv -\frac{1}{3} \sum_{i=1}^{3} \frac{\langle \text{Re} \text{Tr} [U(\frac{1}{T}; \tau) gE_i(\tau, 0) U(\tau; 0) gE_i(0, 0)] \rangle}{\langle \text{Re} \text{Tr} [U(\frac{1}{T}; 0)] \rangle} \]

- using noise reduction techniques to improve signal
- and an Ansatz for the spectral function

→ first continuum estimate for the Heavy Quark Momentum Diffusion Coefficient \( \kappa \)

More detailed analysis of the systematic uncertainties needed
- Different Ansätze for the spectral function
- Other techniques to extract the spectral function

Other Transport coefficients from Effective Field Theories?
NNLO gives more contribution at small and large distances.
NNLO gives more contribution at small and large distances, but some contribution at intermediate distance/frequency still missing

→ improve the model spf or use more clever techniques to extract spf
from Maximum Entropy Method analysis on a fine but finite lattice:

statistical error band from Jackknife analysis

no clear signal for bound states at and above 1.46 $T_c$

study of the continuum limit and quark mass dependence on the way!
Charmonium Spectral function – Transport Peak

\[ D = \frac{\pi}{3 \chi_{00}} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega, \vec{p} = 0, T)}{\omega T} \]

Perturbative estimate (\( \alpha_s \sim 0.2, \ g \sim 1.6 \)):

- LO: \( 2\pi TD \simeq 71.2 \)
- NLO: \( 2\pi TD \simeq 8.4 \)


Strong coupling limit:

\( 2\pi TD = 1 \)

[H.T.Ding, OK et al., PRD86(2012)014509]

Still large systematic uncertainties

- how to extract the spectral function
- cut-off effects become larger with increasing $m_q$
- quark mass dependence → bottomonium
- continuum limit needed

Is there a better observable that is more sensitive to transport properties?