Approach to equilibrium
in weakly coupled nonabelian plasmas

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1401.3751 with M. Abraao York, E. Lu, and G. Moore (McGill)
1405.6318 with E. Lu
1107.5050, 1108.4684, 1209.4091, 1207.1663 with Moore

- **What**: Thermalization in $\alpha \ll 1$ nonabelian gauge theory
- **How**: Using combination of classical field theory and kinetic theory
- **New**: Smooth shift of d.o.f from fields to particles,
  first numerical estimates of bottom-up thermalization
Motivation: Bottom-up thermalization

Anisotropy: $-\log\left(\frac{P_L}{P_T}\right)$

Occupancy:

$f \sim \alpha^{-1}$

$P_L/P_T \sim \alpha^{1/2}$

$f \sim 1$

Thermal

Underoccupied

Radiational breakup $Q_t \sim \alpha^{-13/5}$

Overoccupied

Initial condition $Q_t \sim 1$

Occupancy

- CGC: Initial condition overoccupied

$f(Q) \sim 1/\alpha$

- Expansion makes system underoccupied before thermalizing

$f(Q) \ll 1$

Baier et. al hep-ph/0009237, AK, Moore 1108.4684
Motivation: Bottom-up thermalization

Anisotropy: $-\log\left(\frac{P_L}{P_T}\right)$

Occupancy: $f \sim \alpha^{-1}$

Thermal: $P_L/P_T \sim \alpha^{1/2}$

Underoccupied Overoccupied

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Initial condition $Q_t \sim 1$

Degrees of freedom:
- Overoccupied: Classical field theory, $f \gg 1$
- Underoccupied: (Semi-)classical particles, eff. kinetic theory, $f \ll 1/\alpha$

Full description: Need change of d.o.f. from fields to particles

Overlapping domain of validity $1 \ll f \ll 1/\alpha$: Field-particle duality

Talk by Schlichting
Effective kinetic theory of Arnold, Moore and Yaffe hep-ph/0209353

\[ \frac{df}{dt} = -C_{2\leftrightarrow2}[f] - C_{1\leftrightarrow2}[f] \]

- Soft and collinear divergences lead to nontrivial matrix elements
  - soft: screening, Hard-loop; collinear: LPM, ladder resum
- No free parameters; LO accurate in the \( \alpha \rightarrow 0, \alpha f \rightarrow 0 \) limit.
- Used (in linearized form) e.g. for LO transport coefficients in QCD

Arnold, Moore, Yaffe hep-ph/0302165
Outline:

- Isotropic overoccupied system, field-particle duality
- Isotropic underoccupied system, radiational breakup
- Application to Bottom-up of BMSS
What happens if you have too many soft gluons, $f \sim 1/\alpha$.
No longitudinal expansion.

\[ \ln(f) \]

Initial condition

\[ (e^{\beta p} - 1)^{-1} \]

Thermal

$\sim 1$
What happens if you have **too many soft gluons**, \( f \sim 1/\alpha \).

No longitudinal expansion.

**Initial condition**

**Self-similar cascade**

\[ p_{\text{max}} \sim t^{1/7} \]

\[ f(p_{\text{max}}) \sim t^{-4/7} \]

**Thermal**

\[ (e^{\beta p} - 1)^{-1} \]

\[ f \sim 1 \]
Overoccupied cascade

Lattice and Kinetic Thy. Compared

Form of cascade from classical lattice simulation,

\[ 1 \ll f \lesssim \frac{1}{\alpha} \]

Large-volume: \((Q_a)=0.2, (Q_L)=51.2\), Cont. extr.: down to \((Q_a)=0.1, (Q_L)=25.6\), \(Q_t=2000\), \(\bar{m} = 0.08\)
Overoccupied cascade

Same system, very different degrees of freedom

\[ 1 \lesssim f \ll 1/\alpha \]

Sensitive to the details of the collision terms
Overoccupied cascade

Thermal equilibrium reached once $f \sim 1$ (or $t \sim \frac{1}{\alpha^2 T}$).
Overoccupied cascade

\[ m^2 = \lambda \int_p \frac{f(p)}{p} \]

\[ T_* = \frac{\lambda}{2} \int_p f(p) \left[ 1 + f(p) \right] / m^2 \]

\[ \langle p \rangle = \frac{1}{n} \int_p p f(p) \]

Therm. time through the approach of \( \langle p \rangle - \langle p \rangle_T \sim \exp(-t/t_{eq}) \)

\[ t_{eq} \approx \frac{72.}{1 + 0.12 \log \lambda^{-1}} \frac{1}{\lambda^2 T}, \quad \lambda = 4\pi N_c \alpha. \]
Underoccupied cascade

Isotropic, underoccupied initial conditions, initial scale $\langle p^2 \rangle = Q^2$

Thermalization time parametrically given by stopping time of jet of momentum $Q$:

$$t_{eq} \approx \left( \frac{Q}{T} \right)^{1/2} \frac{1}{\lambda^2 T}$$
Underoccupied cascade

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Underoccupied cascade

Scaling analysis with gaussian and step-cutoff initial conditions

\[
\begin{align*}
\langle p \rangle / \langle p \rangle_T &= \frac{1}{\sqrt{2\pi \langle p \rangle^2}} \\
\langle p^2 \rangle^{1/2} / \langle p \rangle_T &= \frac{1}{\sqrt{2\pi \langle p \rangle^2}} \\
\end{align*}
\]

\[
\lambda = 0.1, 1, 10
\]

\[
T_*/T
\]

\[
\frac{m}{\lambda^{1/2} T}
\]

\[
\begin{align*}
t_{eq} &\approx 34. + 21. \ln\left(\frac{Q}{T}\right) \\
& \quad \times \frac{1}{1 + 0.037 \log \lambda^{-1}} \left(\frac{Q}{T}\right)^{1/2} \frac{1}{\lambda^2 T}
\end{align*}
\]
Connection to heavy-ion physics

Bottom-up thermalization a la BMSS:

- Underoccupied cascade, but expansion reduces the target temperature

\[ \tau_{\text{eq}} \sim \frac{1}{\alpha^2 T} \left( \frac{Q}{T} \right)^{1/2}, \quad \epsilon \sim T^4 \sim \frac{Q^4}{\alpha(Qt)} \Rightarrow Qt \sim \alpha^{-13/5} \]

- Rough estimate: replace parametric estimate by the numerical
  - Estimate for energy density \( \epsilon \approx 1.5 d_A Q^4 / \pi \lambda(Qt) \) and \( \alpha = 0.3 \)

\[ Qt_{\text{eq}} \approx 1.5 \]

Caveats: Angular dependence, no fermions, definition dependence of \( t_{\text{eq}} \), extrapolation to \( \alpha = 0.3 \)
Connection to heavy-ion physics

Quantifying uncertainties:

- For $\alpha = 0.2$:
  \[ Q t_{eq} \approx 4.0 \]

- Varying $\epsilon$ by a factor of 2:
  \[ Q t_{eq} \approx 2.5 \]

- with $\alpha = 0.2$: $Q t_{eq} \approx 8.0$

- Replacing free streaming $(Q t)^{-1}$ by redshifting $(Q t)^{-4/3}$:
  \[ Q t_{eq} < 4 \]

For $Q_s \approx 2 GeV$, corresponds to

\[ t_{eq} \approx 0.1 - 1 fm/c \]
Conclusions

- Combination of classical simulations and effective kinetic theory allows to follow the time evolution from highly occupied initial condition to thermal equilibrium.
- Thermalization times for simple systems faster than naively expected.
- Inserting the underoccupied thermalization time to bottom-up thermalization yields a rough estimate for heavy-ion collisions

\[ t_{\text{eq}} \lesssim 0.1 \text{ - } 1 \text{fm/c} \]

Outlook

- Proper treatment of expansion and angular dependence.
- Implementation of fermions to kinetic theory.
- Inclusions of plasma unstable modes.
- NLO not inconceivable.
- Applications to jets.

p.s. No sign of BEC
Scaling analysis

\[ f_{\text{step}}(p) \propto \Theta(Q_s - p), \quad f_g(p) \propto \exp \left[ -\frac{(Q_s - p)^2}{(Q_s/10)^2} \right] \]

\[ Q^2 \equiv \int_p f(p)p^2 / \int_p f(p) \]

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<th>( n_H/n_T )</th>
<th>( \lambda )</th>
<th>init</th>
<th>run</th>
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In reality many more simulations with varying parameters
Power law from the cascade

- Low scales have time to thermalize: $1/p$
- Non-trivial turbulent Kolmogorov cascade $1/p^{4/3}$, (BEC: $1/p^{3/2}$)?

\[ f \sim p^{-4/3} \]
\[ f \sim p^{-1} \]

\[ m_D = 0.4, Q = 8 \]
\[ m_D = 0.2, Q = 76 \]
\[ m_D = 0.1, Q = 820 \]
\[ m_D = 0.05, Q = 8000 \]
\[ m_D = 0.025, Q = 81000 \]
\[ m_D = 0.0125, Q = 890000 \]