

# Wigner functional of fermion fields

Stanisław Mrówczyński

*Jan Kochanowski University, Kielce, Poland  
& National Centre for Nuclear Research, Warsaw, Poland*

# Motivation – Wigner function

1932 – Eugene Wigner

$$\begin{array}{ccc} \varphi(t, x) & \rightarrow & W(t, x, p) \\ \text{wave function} & & \text{Wigner function} \end{array}$$

$$W(t, x, p) \equiv \int du e^{-ipu} \varphi^*(t, x + u/2) \varphi(t, x - u/2)$$

- ▶  $W(t, x, p) \in \mathbf{R}$
- ▶  $\int dx |\varphi(t, x)|^2 = 1 \quad \Rightarrow \quad \int dx \int \frac{dp}{2\pi} W(t, x, p) = 1$
- ▶  $W(t, x, p) \gtrsim 0$

# Motivation – bosonic Wigner functional

1994 – St. Mrówczyński & B. Müller

$$\begin{array}{ccc} \Phi(t, x) & \rightarrow & W[t, \Phi(x), \Pi(x)] \\ \text{scalar quantum field} & & \text{Wigner functional} \end{array}$$

$$W(t, x, p) \equiv \int du e^{-ipu} \langle x + u/2 | \hat{\rho}(t) | x - u/2 \rangle \quad |x\rangle \text{ – position eigenstate}$$

$$W[t, \Phi(x), \Pi(x)] \equiv \int D\varphi(x) e^{-i \int dx \Pi(x) \varphi(x)} \langle \Phi(x) + \varphi(x)/2 | \hat{\rho}(t) | \Phi(x) - \varphi(x)/2 \rangle$$

$\Phi(x)$  – time-independent scalar field in Schrödinger picture

# Motivation – bosonic Wigner functional

$$\langle O(\hat{\Phi}, \hat{\Pi}) \rangle = \langle O(\Phi, \Pi) \rangle$$

$$\blacktriangleright \langle O(\hat{\Phi}, \hat{\Pi}) \rangle \equiv \frac{1}{\text{Tr}[\hat{\rho}]} \text{Tr}[\hat{\rho} O(\hat{\Phi}, \hat{\Pi})]$$

$$\blacktriangleright \langle O(\Phi, \Pi) \rangle \equiv \frac{1}{Z} \int D\Phi \int \frac{D\Pi}{2\pi} O(\Phi, \Pi) W[t, \Phi, \Pi]$$

$$Z \equiv \int D\Phi \int \frac{D\Pi}{2\pi} W[t, \Phi, \Pi]$$

$W[t, \Phi, \Pi]$  – density in phase-space spanned by  $\Phi$  &  $\Pi$

# Motivation – bosonic Wigner functional

$$i \frac{\partial}{\partial t} \hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)]$$



Semiclassical equation of motion

$$\left[ \frac{\partial}{\partial t} + \int dx \left( \frac{\delta H}{\delta \Pi(x)} \frac{\delta}{\delta \Phi(x)} - \frac{\delta H}{\delta \Phi(x)} \frac{\delta}{\delta \Pi(x)} \right) \right] W[t, \Phi, \Pi] = 0$$

Liouville's equation

# Motivation – bosonic Wigner functional

Free field in thermal equilibrium

$$\hat{\rho} = e^{-\beta\hat{H}}$$



$$W[\Phi, \Pi] \sim \exp\left[-\frac{\beta}{2} \int \frac{dp}{2\pi} \Delta(p) \left( \Pi^*(p)\Pi(p) + (p^2 + m^2)\Phi^*(p)\Phi(p) \right)\right] \approx \exp[-\beta H]$$

classical limit

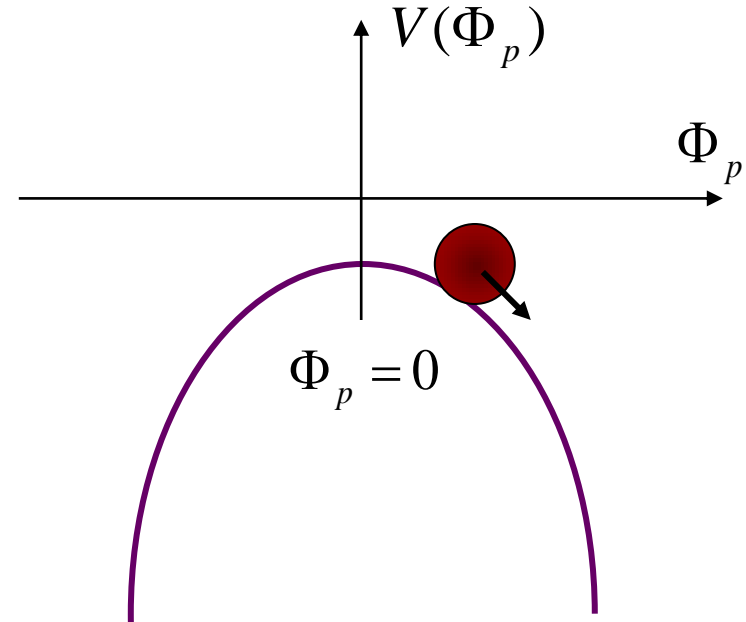
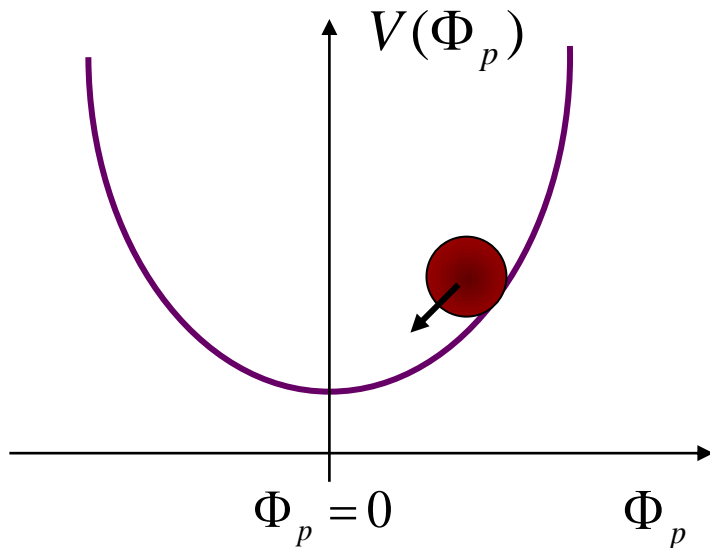
$$\Delta(p) \equiv \frac{2T}{E_p} \tanh\left(\frac{E_p}{2T}\right), \quad E_p \equiv \sqrt{p^2 + m^2}, \quad \beta \equiv \frac{1}{T}$$

# Motivation – bosonic Wigner functional

Simple application – *rollover* phase transition

$$V(\Phi) = -\frac{1}{2} m^2(t) \Phi^2(x), \quad m^2(t) = \begin{cases} m^2 & t < 0 \\ -\mu^2 & t \geq 0 \end{cases}$$

$$p^2 < \mu^2$$



# Definition of fermionic Wigner functional

$$W[t, \Psi(x), \Pi(x)] \equiv \int D\varphi(x) e^{-i\int dx \Pi(x)\varphi(x)} \langle \Psi(x) + \varphi(x)/2 | \hat{\rho}(t) | \Psi(x) - \varphi(x)/2 \rangle$$

Does it make any sense?

$\Psi(x)$ ,  $\Pi(x)$ ,  $\varphi(x)$  - Grassmann (anticommuting) fields

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$$\Psi(x) \rightarrow \{\Psi_1, \Psi_2, \Psi_3, \dots\} \quad \Psi_i \equiv \Psi(x_i)$$

▶  $\Psi_i \Psi_j + \Psi_j \Psi_i = 0$        $\Psi_i^2 = 0$

▶ Berezin integrals:       $\int d\Psi_i = 0$ ,       $\int d\Psi_i \Psi_j = \delta^{ij}$



# Simple example

Harmonic oscillator in a ground state

$$\langle x_1 | \hat{\rho}(t) | x_2 \rangle \sim \exp \left[ -\frac{1}{2} (a x_1^2 + a x_2^2) \right] \quad \Rightarrow \quad W(t, x, p) \sim \exp \left[ -a x^2 - \frac{p^2}{a} \right]$$

$$\langle \Psi_1(x) | \hat{\rho}(t) | \Psi_2(x) \rangle = \exp \left[ -\frac{1}{2} \int dx \left( \Psi_1(x) \mathbf{A} \Psi_1(x) + \Psi_2(x) \mathbf{A} \Psi_2(x) \right) \right]$$

$$\Psi_i = \begin{pmatrix} \chi_i \\ \psi_i \end{pmatrix} \text{ - Weyl spinor}$$

$\mathbf{A}$  - antisymmetric  $2 \times 2$  matrix

$$\mathbf{A}^T = -\mathbf{A}$$

$$W[t, \Psi(x), \Pi(x)] = \exp \left[ -\int dx \left( \Psi(x) \mathbf{A} \Psi(x) - \Pi(x) \mathbf{A}^{-1} \Pi(x) \right) \right]$$

# Fermionic Wigner functional

$$\langle O(\hat{\Psi}, \hat{\Pi}) \rangle = \langle O(\Psi, \Pi) \rangle$$

$$\blacktriangleright \quad \langle O(\hat{\Psi}, \hat{\Pi}) \rangle \equiv \frac{1}{Z} \text{Tr}[\hat{\rho} O(\hat{\Psi}, \hat{\Pi})]$$

$$\blacktriangleright \quad \langle O(\Psi, \Pi) \rangle \equiv \frac{1}{Z} \int D\Psi \int \frac{D\Pi}{2\pi} O(\Psi, \Pi) W[t, \Psi, \Pi]$$

$$Z \equiv \text{Tr}[\hat{\rho}] = \int D\Psi \int \frac{D\Pi}{2\pi} W[t, \Psi, \Pi]$$

$W[t, \Psi, \Pi]$  - density in phase-space spanned by  $\Psi$  &  $\Pi$

# Example of computation

$$Z = \int D\Psi \int \frac{D\Pi}{2\pi} W[t, \Psi, \Pi] = \int D\Psi \int \frac{D\Pi}{2\pi} \int D\varphi(x) e^{-i\int dx \Pi(x)\varphi(x)} \langle \Psi(x) + \varphi(x)/2 | \hat{\rho}(t) | \Psi(x) - \varphi(x)/2 \rangle$$

Discretization

$$\int \frac{D\Pi}{2\pi} e^{-i\int dx \Pi(x)\varphi(x)} = \quad ? \quad \left\{ \begin{array}{l} \varphi(x) \rightarrow \{\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n\}, \quad \varphi_i \equiv \varphi(x_i) \\ \Pi(x) \rightarrow \{\Pi_1, \Pi_2, \Pi_3, \dots, \Pi_n\}, \quad \Pi_i \equiv \Pi(x_i) \end{array} \right.$$

Berezin integrals:  $\int d\Pi_i = 0, \quad \int d\Pi_i \Pi_j = \delta^{ij}$

$$\int \frac{D\Pi_n}{2\pi} \frac{D\Pi_{n-1}}{2\pi} \dots \frac{D\Pi_1}{2\pi} e^{-i\Delta x \sum_{j=1}^n \Pi_j \varphi_j} = (-i\Delta x)^n \int \frac{D\Pi_n}{2\pi} \frac{D\Pi_{n-1}}{2\pi} \dots \frac{D\Pi_1}{2\pi} \left( \sum_{j=1}^n \Pi_j \varphi_j \right)^n$$

## Example of computation cont.

$$\int \frac{D\Pi}{2\pi} e^{-i\int dx \Pi(x)\varphi(x)} = (-i\Delta x)^n \int \frac{D\Pi_n}{2\pi} \frac{D\Pi_{n-1}}{2\pi} \dots \frac{D\Pi_1}{2\pi} \left( \sum_{j=1}^n \Pi_j \varphi_j \right)^n = - \left( -\frac{i\Delta x}{2\pi} \right)^n \varphi_1 \varphi_2 \dots \varphi_n$$

$$\int d\varphi_n d\varphi_{n-1} \dots d\varphi_1 \overbrace{\varphi_1 \varphi_2 \dots \varphi_n}^{\delta(\varphi_1, \varphi_2, \dots, \varphi_n)} f(\varphi_1, \varphi_2, \dots, \varphi_n) = f_0$$

function in n-dimensional Grassmann algebra spanned by  $\{1, \varphi_i, \varphi_i \varphi_j, \dots, \underbrace{\varphi_i \varphi_j \dots \varphi_k}_n\}$

$$f(\varphi_1, \varphi_2, \dots, \varphi_n) = f_0 + \sum_{i=1}^n f_1(i) \varphi_i + \sum_{i,j=1}^n f_2(i, j) \varphi_i \varphi_j + \dots$$

$$\int \frac{D\Pi}{2\pi} e^{-i\int dx \Pi(x)\varphi(x)} \propto \delta[\varphi(x)]$$

## Example of computation cont.

$$Z = \int D\Psi \int \frac{D\Pi}{2\pi} \int D\varphi(x) e^{-i\int dx \Pi(x)\varphi(x)} \langle \Psi(x) + \varphi(x)/2 | \hat{\rho}(t) | \Psi(x) - \varphi(x)/2 \rangle$$



$$\int \frac{D\Pi}{2\pi} e^{-i\int dx \Pi(x)\varphi(x)} \propto \delta[\varphi(x)]$$

$$Z = \int D\Psi \langle \Psi(x) | \hat{\rho}(t) | \Psi(x) \rangle = \text{Tr}[\hat{\rho}]$$

$$\int D\Psi \int \frac{D\Pi}{2\pi} W[t, \Psi, \Pi] = \text{Tr}[\hat{\rho}]$$

# Fermionic Wigner functional

$$i \frac{\partial}{\partial t} \hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)]$$



Equation of motion

$$\left[ \frac{\partial}{\partial t} + \int dx \left( \frac{\delta H}{\delta \Pi(x)} \frac{\delta}{\delta \Psi(x)} - \frac{\delta H}{\delta \Psi(x)} \frac{\delta}{\delta \Pi(x)} \right) \right] W[t, \Psi, \Pi] = 0$$

Exact equation of motion is of the Liouville form

# Conclusions

Although bosonic and fermionic fields are very different,  
the Wigner functionals have very similar properties.

For more: St. Mrówczyński, Physical Review D **87**, 065026 (2013)