Chiral symmetry restoration at strong coupling?

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based on work with

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Outline

- Brief history of the QCD phase diagram ($\langle\bar{\psi}\psi\rangle$ as a function of N_f) at $g = \infty$ and $T = 0$
- Calculating $\langle\bar{\psi}\psi\rangle$ diagrammatically
- Results

Introduction: $\langle\bar{\psi}\psi\rangle$ at $g = \infty$

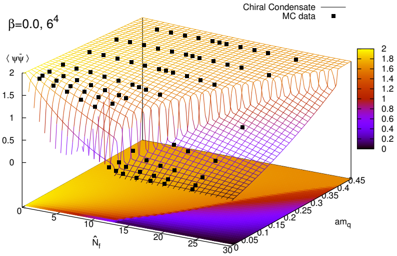
For $N_f = 0$

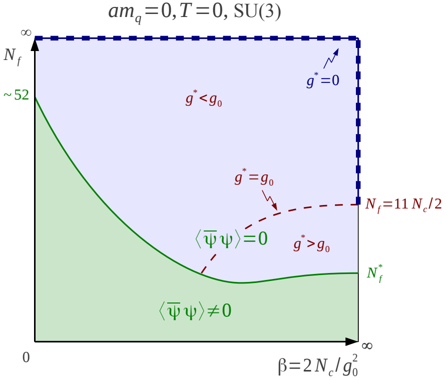
$$\langle\bar{\psi}\psi\rangle \neq 0\,.$$

What happens as N_f is increased?
Could the chiral symmetry be restored as we see from simulations at more moderate coupling strengths?

- Using a $1/d$ expansion to calculate $\langle\bar{\psi}\psi\rangle$ analytically [Kluberg-Stern, Morel, Petersson 1982] find that there is no transition to a phase in which $\langle\bar{\psi}\psi\rangle = 0$ for any N_f
- A mean field analysis based on [Damgaard, Hochberg, Kawamoto 1985] also suggests that the deconfinement critical temperature $T_c \neq 0$ for all N_f
- Using Monte-Carlo simulations [de Forcrand, Kim, Unger 2013] find that a transition does occur, around $N_f \sim 13$ staggered fermion flavours
- Using a diagrammatic approach [Tomboulis 2013] also finds that a transition occurs, around $N_f \sim 10.7$ staggered flavours

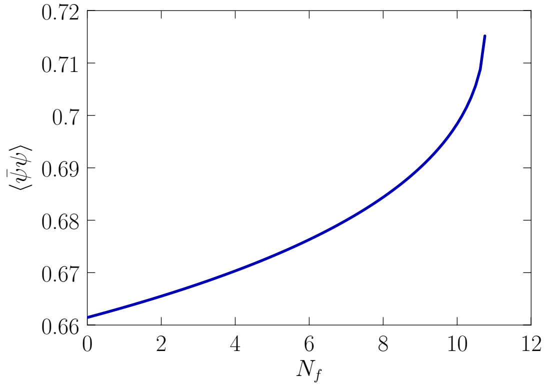
Simulation results [de Forcrand, Kim, Unger 2013]





$N_{fc} \sim 13$ staggered flavours.

Leading order strong coupling expansion [Tomboulis 2013]



$N_{fc} \sim 10.7$ staggered flavours for $N_c = 3$.

NOTE: This is a plot we created using the formulas below from [Tomboulis 2013].

$$\langle\bar{\psi}\psi\rangle = -\lim_{m\to 0} \mathrm{tr}\, G$$

$$\mathrm{tr}\, G = \left[m - \left(4d^2(d-1)\frac{N_f}{N_c}\left(\frac{g(m)}{2}\right)^9 - d\left(\frac{g(m)}{2}\right)\right)\right]^{-1}$$

$$g(m) = \left[m - \left(4d^2(d-1)\frac{N_f}{N_c}\left(\frac{g(m)}{2}\right)^9 - \left(\frac{2d-1}{2}\right)\left(\frac{g(m)}{2}\right)\right)\right]^{-1}$$

Calculating the chiral condensate

[Blairon, Brout, Englert and Greensite (1981); Martin and Siu (1983); Tomboulis (2013)]

The chiral condensate is obtained from

$$\langle\bar{\psi}(x)\psi(x)\rangle = -\mathrm{tr}\left[G(x,x)\right] = -\frac{1}{N_f}\partial_m \log Z\,.$$

Integrating out the fermion contribution results in

$$G(x,x) = \frac{\int \mathcal{D}U \det\left[1+K^{-1}M(U)\right]\left[\left[1+K^{-1}M(U)\right]^{-1}K^{-1}\right]_{xx}}{\int \mathcal{D}U \det\left[1+K^{-1}M(U)\right]}\,,$$

with

$$M_{xy} \equiv \frac{1}{2}\left[\gamma_\mu U_\mu(x)\delta_{y,x+\hat{\mu}} - \gamma_\mu U_\mu^\dagger(x-\hat{\mu})\delta_{y,x-\hat{\mu}}\right]\,,$$

$$K^{-1}_{xy} = m^{-1}\mathbb{I}_{N_f}\mathbb{I}_{N_c}\delta_{xy}\,.$$

The $K^{-1} \sim \frac{1}{m}$ suggests performing a hopping expansion.

Hopping expansion

Performing a hopping expansion on the fermion determinant leads to

$$\det\left[1+K^{-1}M\right] = \exp \mathrm{tr}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(K^{-1}M)^n\right] ,$$

which is a sum over closed loops.

Performing a hopping expansion on the contribution from the 2-pt correlator results in

$$\left[\left[1+K^{-1}M\right]^{-1}K^{-1}\right]_{xx} = \frac{1}{m}\left[\sum_{n=0}^{\infty}(-1)^n(K^{-1}M)^n\right]_{xx} .$$

which contains all loops that begin and end at site x.

Since $\mathrm{tr}\left[\text{odd } \# \text{ of } \gamma_\mu\text{'s}\right]=0$, only contributions with n even contribute. For example, for $n=2$

$$\left[(K^{-1}M)^2\right]_{xx} = \frac{1}{(2m)^2}\sum_{\mu,\nu}\sum_{y}\left[\gamma_\mu\gamma_\nu\right]\left[U_\mu(x)\delta_{y,x+\hat{\mu}} - U_\mu^\dagger(x-\hat{\mu})\delta_{y,x-\hat{\mu}}\right] \times \left[U_\nu(y)\delta_{x,y+\hat{\nu}} - U_\nu^\dagger(y-\hat{\nu})\delta_{x,y-\hat{\nu}}\right] .$$

Extending Martin and Siu

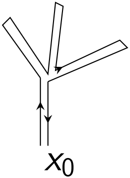
In general the chiral condensate takes the form

$$\frac{\mathrm{tr}[G(x,x)]}{N_s N_f d_R} = \frac{1}{m}\sum_{L=0}^{\infty}(-1)^L \frac{A(L)}{(2m)^{2L}},$$

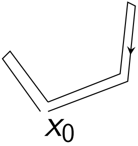
where $A(L)$ is the contribution of all diagrams with $2L$ links which start and end at $x = x_0$.

A general graph can be built out of irreducible graphs $I(l)$ of $2l$ links.

Irreducible

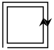


Reducible

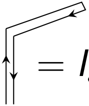


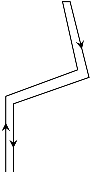
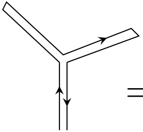
Irreducible graphs cannot be separated into smaller segments which start and end at x_0.

Irreducible diagrams

Irreducible graphs are built iteratively out of all possible combinations of smaller segments attached to a "base diagram" a) , or b) , or ... .

$I(1) =$  $= I_a(1) = 2d$

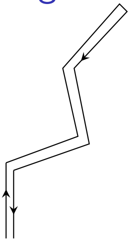
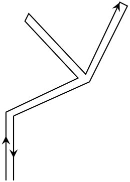
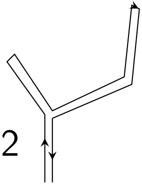
$I(2) =$  $= I_a(2) = 2d \ [I_a(1) \ a_0']$

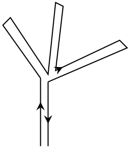
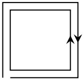
$I(3) =$  $+$  $= I_a(3)$

$$= 2d \left[I_a(2) a_0' + I_a(1)^2 a_0'^2 \right]$$

with $a_0' = \frac{2d-1}{2d}$.

Irreducible diagrams

$I(4) =$  $+$  $+\ 2$ 

$+$  $+$ 

$$= I_a(4) + I_b(4)$$

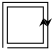
$$= 2d\left[I_a(3)a_0' + 2I_a(1)I_a(2)a_0'^2 + I_a(1)^3 a_0'^3\right] - 4d(d-1)\frac{N_f}{N_c}$$

...

General diagrams

To obtain the contribution of all general diagrams $A(L)$ of a length $2L$, take all combinations of irreducible bits.

$$A(L) = \sum_{l=1}^{L} I(l)A(L-l)\,, \qquad L \geq 1\,; \qquad A(0) = 1\,,$$

where the irreducible graphs can begin with a) , or b) , or ... .

$$I(L) = 2dF_0(L-1) - 4d(d-1)\frac{N_f}{N_c}F_1(L-4)^7 + \ldots\,.$$

with $I(0) = 0$. $F_n(L)$ represents all possible graphs of length $2L$ which start and end on a site on a base diagram of area n.

$$F_n(L) = \sum_{\substack{l_i=1,2,\ldots,\\ k_j=4,8,\ldots,\\ \sum l_i+k_j=L-1}} I_a(l_1)I_a(l_2)\ldots I_a(l_p)I_b(k_1)I_b(k_2)\ldots I_b(k_q)a_n'^p b_n'^q\,,$$

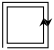
with $F_n(0) = 1$. $x_n' \equiv \frac{x_n}{d_x}$.

For example: $$a_0' = \frac{2d-1}{2d}\,, \qquad b_0' = \frac{4(d-1)^2}{4d(d-1)}\,.$$

Generating all irreducible graphs

The generating function for irreducible graphs, which gives the total contribution of all irreducible graphs including the mass dependence, is

$$W_I = \sum_{l=0}^{\infty} \left(-\frac{1}{4m^2}\right)^l I(l) = W_a + W_b + \ldots ,$$

where W_a is all irreducible graphs starting with . W_b is all irreducible graphs starting with , etc. These take the form

$$W_a = 2dx \sum_{n=0}^{\infty} [a_0' W_a + b_0' W_b + \ldots]^n = \frac{2dx}{1 - a_0' W_a - b_0' W_b - \ldots} ,$$

$$W_b = -4d(d-1)\frac{N_f}{N_c}x^4 \left[\sum_{n=0}^{\infty} [a_1' W_a + b_1' W_b + \ldots]^n\right]^7 = \frac{-4d(d-1)\frac{N_f}{N_c}x^4}{(1 - a_1' W_a - b_1' W_b - \ldots)^7} ,$$

$$\ldots .$$

with $x \equiv -\frac{1}{4m^2}$. The chiral condensate is then obtained from

$$\frac{\mathrm{tr}[G(x,x)]}{N_s N_f N_c} = \lim_{m \to 0} \frac{1}{m} \left(\frac{1}{1 - W_I}\right) .$$

Chiral limit $m \to 0$

To work directly in the massless limit it is convenient to introduce the variables $g_x \equiv -\frac{2mW_x}{d_x}$,

$$g \equiv d_a g_a + d_b g_b + \dots .$$

Taking $m \to 0$, the system of equations

$$g_a = \frac{1}{a_0 g_a + b_0 g_b + \dots},$$

$$g_b = \frac{\frac{N_f}{N_c}}{(a_1 g_a + b_1 g_b + \dots)^7},$$

$$g_c = \frac{\frac{N_f}{N_c}}{(a_2 g_a + b_2 g_b + \dots)^{11}},$$

$$\cdots,$$

can be solved numerically. The chiral condensate is then obtained from

$$\frac{\mathrm{tr}[G(x,x)]}{N_s N_f N_c} = \frac{2}{g}.$$

Calculating fundamental diagrams

To obtain the total contribution of a diagram, one must include the following

- A factor $\frac{1}{i!}(-N_f N_s)^i$, for a number of overlapping closed internal loops i,
- A mass factor $\left(-\frac{1}{4m^2}\right)^n$, for n pairs of links,
- $(-1)^k$ for k permutations of γ matrices,
- [...], containing the result obtained by performing the group integrations,
- {...}, containing the dimensionality of the graph.

Group integrals [Creutz, Cvitanovic]

Group integrals for overlapping links of the form  , 
are nonzero $\forall\, N_c \equiv N$.

$$\int_{SU(N)} \mathrm{d}U \; U_a{}^d U^{\dagger\, b}_c = \frac{1}{N} \delta^d_c \delta^b_a \,,$$

$$\begin{aligned} \int_{SU(N)} \mathrm{d}U \; U^{\dagger\, a}_h U^{\dagger\, b}_g U_c{}^f U_d{}^e = \; & \frac{1}{2N(N+1)} \left(\delta^a_d \delta^b_c + \delta^a_c \delta^b_d \right) \left(\delta^e_h \delta^f_g + \delta^e_g \delta^f_h \right) \\ & + \frac{1}{2N(N-1)} \left(\delta^a_d \delta^b_c - \delta^a_c \delta^b_d \right) \left(\delta^e_h \delta^f_g - \delta^e_g \delta^f_h \right) \,. \end{aligned}$$

The group integral of  is nonzero for $SU(3)$

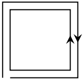
$$\int_{SU(3)} \mathrm{d}U \; U_i^j U_k{}^l U_m{}^n = \frac{1}{6} \epsilon_{ikm} \epsilon^{jln} \,.$$

Fundamental diagrams $L = 2, 4, 6$

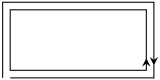
$L = 2$

 $= -\frac{1}{4m^2}\{2d\}$

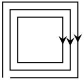
$L = 4$

 $= \left(-\frac{1}{4m^2}\right)^4 (-1)^2(-N_f)\left[\frac{1}{N_c}\right]\{4d(d-1)\}$

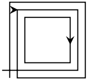
$L = 6$

 $= \left(-\frac{1}{4m^2}\right)^6 (-N_f)\left[\frac{1}{N_c}\right]\{12d(d-1)(2d-3)\}$

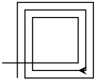
Fundamental diagrams $L = 6,\ N_c = 3$



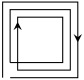
$$= \frac{1}{2!}\left(-\frac{1}{4m^2}\right)^6 (-1)^3(-N_f)^2 \left[\frac{1}{3}\right] \{4d(d-1)\}$$



$$= \left(-\frac{1}{4m^2}\right)^6 (-1)^3(-N_f) \left[-\frac{1}{3}\right] \{4d(d-1)\}$$



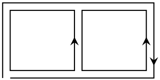
$$= \left(-\frac{1}{4m^2}\right)^6 (-1)^3 \left[\frac{1}{3}\right] \{4d(d-1)\}$$



$$= \left(-\frac{1}{4m^2}\right)^6 (-1)^3(-N_f) \left[-\frac{1}{3}\right] \{4d(d-1)\}$$

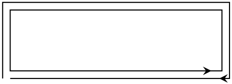
Fundamental diagrams $L = 7, 8$

$L = 7$

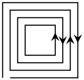


$$= \frac{1}{2!}\left(-\frac{1}{4m^2}\right)^7 (-1)^2(-N_f)^2 \left[\frac{1}{N_c^2}\right] \{12d(d-1)(2d-3)\}$$

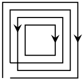
$L = 8$



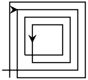
$$= \left(-\frac{1}{4m^2}\right)^8 (-1)^2(-N_f) \left[\frac{1}{N_c}\right] \{36d(d-1)(2d-3)^2\}$$



$$= \frac{3}{3!}\left(-\frac{1}{4m^2}\right)^8 (-1)^4(-N_f)^3 \left[\frac{2}{N_c}\right] \{4d(d-1)\}$$



$$= \frac{2}{2!}\left(-\frac{1}{4m^2}\right)^8 (-1)^4(-N_f)^2 [0] \{4d(d-1)\}$$



$$= \left(-\frac{1}{4m^2}\right)^8 (-1)^4(-N_f) \left[\frac{2}{N_c}\right] \{4d(d-1)\}$$

Fundamental diagrams $L > 9$

To obtain diagrams for $L > 9$ we need additional group integrals.

For example, to get to $L = 16$ for the fundamental and $N_c = 3$ we would need

$$\int_{SU(3)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^{\dagger j} \,,$$

$$\int_{SU(N)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g^{\dagger h} U_i^{\dagger j} U_k^{\dagger l} \,,$$

$$\int_{SU(3)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^j U_k{}^l \,,$$

$$\int_{SU(3)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^j U_k^{\dagger l} U_m^{\dagger}{}^n \,,$$

$$\int_{SU(N)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^{\dagger j} U_k^{\dagger l} U_m^{\dagger}{}^n U_o^{\dagger p} \,.$$

Note that each of the three $SU(3)$ integrals can be transformed into one of the $SU(N)$ integrals and Levi-Cevita tensors.

Fundamental diagrams $L > 9$

$$\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^{\dagger j} = \frac{1}{2}\epsilon_{gmn}\epsilon^{hkl} \boxed{\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_k^{\dagger m} U_l^{\dagger n} U_i^{\dagger j}}\,,$$

$$\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^j U_k{}^l$$

$$= \frac{1}{4}\epsilon_{im_1n_1}\epsilon^{ja_1b_1}\epsilon_{km_2n_2}\epsilon^{la_2b_2} \boxed{\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_{a1}^{\dagger m_1} U_{b1}^{\dagger n_1} U_{a2}^{\dagger m_2} U_{b2}^{\dagger n_2}}\,,$$

$$\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^j U_k^{\dagger l} U_m^{\dagger\ n}$$

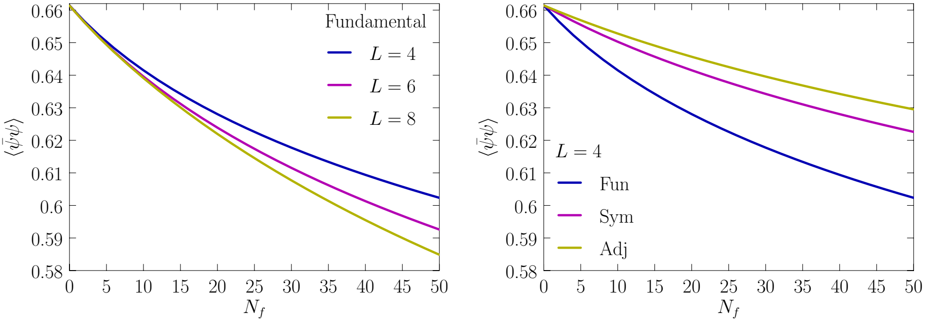
$$= \frac{1}{2}\epsilon_{iab}\epsilon^{jcd} \boxed{\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_c^{\dagger a} U_d^{\dagger b} U_k^{\dagger l} U_m^{\dagger\ n}}\,.$$

using [for $SU(3)$]

$$U_i^j = \frac{1}{2}\epsilon_{imn}\epsilon^{jkl} U_k^{\dagger m} U_l^{\dagger n}\,,$$

$$U_i^{\dagger j} = \frac{1}{2}\epsilon_{imn}\epsilon^{jkl} U_k{}^m U_l{}^n\,.$$

PRELIMINARY Results



Chiral condensate (normalised by $\frac{1}{d_R}$) for $SU(3)$ including ONLY area $n = 0$ and $n = 1$ diagrams

Conclusions and outlook

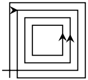
- We calculated the chiral condensate at $g = \infty$ for QCD with N_f flavours using a truncated diagrammatic expansion and find that $\langle\bar{\psi}\psi\rangle \neq 0$ at all N_f, though it approaches zero as $N_f \to \infty$.
- The expansion appears to converge for area $n = 0$ and $n = 1$ diagrams
- We calculated group integrals including up to 4 U's and 4 $U^\dagger$'s using the technique of Young projectors, which can be used to calculate diagrams up to $L = 8$ in the fundamental and $L = 4$ in the adjoint, symmetric, and antisymmetric.
- Area $n > 1$ diagrams have been calculated up to $L = 8$ but still need to be included in the calculation of the chiral condensate.

Backup slides

Issue: “diagram overlap problem”

More often than not, overlapping diagrams with nonzero area $(n > 0)$ are miscounted.

$L = 8$



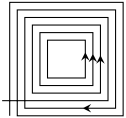
$$= \boxed{\frac{1}{2!}} \left(-\tfrac{1}{4m^2}\right)^{16} (-1)^4(-N_f)^2 \boxed{[0]}\,,$$

however, it gets counted as

$$\left(-\frac{1}{4m^2}\right)^{16} (-1)^4(-N_f)^2 \left[\frac{1}{N_c^2}\right] .$$

Issue: "diagram overlap problem"

$L = 12$

 $= \left(-\frac{1}{4m^2}\right)^{24}(-1)^6(-N_f)^3\,\boxed{[0]}\,,$

for $N_c \geq 3$. For $N_c = 2$ the result is $\left(-\frac{1}{4m^2}\right)^{24}(-1)^6(-N_f)^3\left[-\frac{1}{2}\right]$.

In either case it gets counted as

$$\left(-\frac{1}{4m^2}\right)^{24}(-1)^6(-N_f)^3\left[\frac{1}{N_c^3}\right]\,.$$

One can account for mis-counting at each order in L in which it appears (starting at $L = 8$).

Group integration with Young Projectors

All integrals we need can be converted to the form

$$\int_{SU(N)} \mathrm{d}U \; U_{\alpha_1}{}^{\beta_1} \ldots U_{\alpha_n}{}^{\beta_n} (U^\dagger)_{\gamma_1}{}^{\delta_1} \ldots (U^\dagger)_{\gamma_n}{}^{\delta_n}$$

Calculating the direct product of n U's ($U^\dagger$'s) leads to a direct sum of representations R (S).

The integral can be obtained from the Young Projectors $\mathbb{P}$ of these representations using

$$\int_{SU(N)} \mathrm{d}U \; R_a{}^b (S^\dagger)_c{}^d = \frac{1}{d_R} (\mathbb{P}^R)_a{}^d (\mathbb{P}^S)_c{}^b \; \delta_{RS} \, .$$

Young projectors $\mathbb{P}$

Consider for example the integral

$$I_2 \equiv \int_{\mathrm{SU}(N)} dU \, U_{\alpha_1}{}^{\beta_1} U_{\alpha_2}{}^{\beta_2} (U^\dagger)_{\gamma_1}{}^{\delta_1} (U^\dagger)_{\gamma_2}{}^{\delta_2} \, .$$

The direct product $\mathbf{N} \otimes \mathbf{N}$ is

$$\boxed{\alpha_1} \otimes \boxed{\alpha_2} = \boxed{\alpha_1}\boxed{\alpha_2} \oplus \begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline \end{array} .$$

The Young projectors are thus formed by symmetrising, and antisymmetrising in α_1 and α_2,

$$\mathbb{P}^{S}_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \frac{1}{2}\left(\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} + \delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2}\right) , \qquad \mathbb{P}^{AS}_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \frac{1}{2}\left(\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} - \delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2}\right) .$$

The resulting integral is

$$I_2 = \frac{2}{N(N+1)} \mathbb{P}^{S}_{\alpha_1\alpha_2}{}^{\delta_1\delta_2} \mathbb{P}^{S}_{\gamma_1\gamma_2}{}^{\beta_1\beta_2} + \frac{2}{N(N-1)} \mathbb{P}^{AS}_{\alpha_1\alpha_2}{}^{\delta_1\delta_2} \mathbb{P}^{AS}_{\gamma_1\gamma_2}{}^{\beta_1\beta_2} \, .$$

Additional group integrals

$$I_3 \equiv \int_{\mathrm{SU}(N)} dU\, U_{\alpha_1}{}^{\beta_1} U_{\alpha_2}{}^{\beta_2} U_{\alpha_3}{}^{\beta_3} (U^\dagger)_{\gamma_1}{}^{\delta_1} (U^\dagger)_{\gamma_2}{}^{\delta_2} (U^\dagger)_{\gamma_3}{}^{\delta_3} \,.$$

with group decomposition

$$\boxed{\alpha_1}\otimes\boxed{\alpha_2}\otimes\boxed{\alpha_3} = \begin{array}{|c|c|c|}\hline \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \end{array}\,(S)\oplus \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,(M)\oplus \begin{array}{|c|c|}\hline \alpha_1 & \alpha_3 \\ \hline \alpha_2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,(\tilde{M})\oplus \begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline \alpha_3 \\ \hline \end{array}\,(AS)\,,$$

results in

$$\begin{aligned} I_3 = {} & \\ & \frac{6}{N(N+1)(N+2)} \mathbb{P}^{S}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_2\delta_3} \mathbb{P}^{S}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_2\beta_3} + \frac{3}{N(N^2-1)} \mathbb{P}^{M}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_2\delta_3} \mathbb{P}^{M}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_2\beta_3} \\ & + \frac{3}{N(N^2-1)} \mathbb{P}^{\tilde{M}}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_2\delta_3} \mathbb{P}^{\tilde{M}}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_2\beta_3} + \frac{3}{N(N^2-1)} \mathbb{P}^{M}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_3\delta_2} \mathbb{P}^{\tilde{M}}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_3\beta_2} \\ & + \frac{3}{N(N^2-1)} \mathbb{P}^{\tilde{M}}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_3\delta_2} \mathbb{P}^{M}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_3\beta_2} + \frac{6}{N(N-1)(N-2)} \mathbb{P}^{AS}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_2\delta_3} \mathbb{P}^{AS}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_2\beta_3} \,. \end{aligned}$$

Higher dimensional representations

Higher dimensional representations can be written in terms of the fundamental and anti-fundamental. For example,

Symmetric

$$(U^S)_a{}^b = (U^S)_{(\alpha_1\alpha_2)}{}^{(\beta_1\beta_2)} = (\mathbb{P}^S)_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2} U_{\gamma_1}{}^{\delta_1} U_{\gamma_2}{}^{\delta_2} (\mathbb{P}^S)_{\delta_1\delta_2}{}^{\beta_1\beta_2}$$
$$= \frac{1}{2}\left(U_{\alpha_1}{}^{\beta_1} U_{\alpha_2}{}^{\beta_2} + U_{\alpha_1}{}^{\beta_2} U_{\alpha_2}{}^{\beta_1}\right)$$

$a, b = 1, ..., d_S$.

Antisymmetric

$$(U^{AS})_m{}^n = (U^{AS})_{[\alpha_1\alpha_2]}{}^{[\beta_1\beta_2]} = (\mathbb{P}^{AS})_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2} U_{\gamma_1}{}^{\delta_1} U_{\gamma_2}{}^{\delta_2} (\mathbb{P}^{AS})_{\delta_1\delta_2}{}^{\beta_1\beta_2}$$
$$= \frac{1}{2}\left(U_{\alpha_1}{}^{\beta_1} U_{\alpha_2}{}^{\beta_2} - U_{\alpha_1}{}^{\beta_2} U_{\alpha_2}{}^{\beta_1}\right)$$

$m, n = 1, ..., d_{AS}$.

Higher dimensional representations

Adjoint

$$(U^A)_a{}^b = 2\,\mathrm{Tr}\left(U t_a U^\dagger t^b\right) \,,$$

where the t_a are fundamental generators of $SU(N)$ satisfying

$$\mathrm{Tr}\,(t_a t_b) = \frac{1}{2}\delta_{ab}\,.$$

At leading order it is sufficient to use

$$\int_{SU(N)} \mathrm{d}U \; (U^R)_a{}^b (U^{R\dagger})_c{}^d = \frac{1}{d_R}\delta_a{}^d \delta_c{}^b\,.$$

$$\int_{SU(N)} \mathrm{d}U \; (U^A)_a{}^b (U^A)_c{}^d = \frac{1}{d_A}\delta_{ac}\delta^{bd}\,.$$

At the next order in the adjoint it is necessary to consider 3-link integrals.

3-link adjoint integrals

We are interested in integrals of the form

$$I_n^A \equiv \int dU\,(U^A)_{a_1}{}^{b_1}\cdots(U^A)_{a_n}{}^{b_n}$$
$$= 2^n\,(t_{a_1})_{\beta_1}{}^{\gamma_1}(t^{b_1})_{\delta_1}{}^{\alpha_1}\cdots(t_{a_n})_{\beta_n}{}^{\gamma_n}(t_{b_n})_{\delta_n}{}^{\alpha_n}\int dU\,U_{\alpha_1}{}^{\beta_1}\cdots U_{\alpha_n}{}^{\beta_n}U^{\dagger}_{\gamma_1}{}^{\delta_1}\cdots U^{\dagger}_{\gamma_n}{}^{\delta_n}$$

For example, for $n = 3$, plugging in the result for the fundamental integral and simplifying using the identity

$$t_a t_b = \frac{1}{2N}\delta_{ab}\mathbf{1}_N + \frac{1}{2}d_{abc}t_c + \frac{i}{2}f_{abc}t_c\,,$$

results in

$$I_3^A = \frac{N}{(N^2-1)(N^2-4)}d_{a_1a_2a_3}d^{b_1b_2b_3} + \frac{1}{N(N^2-1)}f_{a_1a_2a_3}f^{b_1b_2b_3}\,.$$

where

$$if_{abc} = 2\,\mathrm{Tr}\left([t_a,t_b]t_c\right)\,,$$
$$d_{abc} = 2\,\mathrm{Tr}\left(\{t_a,t_b\}t_c\right)\,.$$

Bars and Green integrals [Bars and Green 1979]

Bars and Green calculate integrals of the form

$$F_n \equiv \int_{SU(N)} \mathrm{d}U \; [\mathrm{tr}(AU)]^n [\mathrm{tr}(A^\dagger U^\dagger)]^n$$

$$= \sum_{\substack{i_1,\ldots,i_n,\\ j_1,\ldots,j_n,\\ k_1,\ldots,k_n,\\ l_1,\ldots,l_n}} A_{i_1}{}^{j_1} \ldots A_{i_n}{}^{j_n} (A^\dagger)_{k_1}{}^{l_1} \ldots (A^\dagger)_{k_n}{}^{l_n} \int_{\mathrm{SU}(N)} \mathrm{d}U \, U_{j_1}{}^{i_1} \ldots U_{j_n}{}^{i_n} (U^\dagger)_{l_1}{}^{k_1} \ldots (U^\dagger)_{l_n}{}^{k_n}$$

This integral is a generating function for the types of integrals we are interested in.

One can obtain our integrals by separating out the $A_{i_1}{}^{j_1} \cdots A_{i_n}{}^{j_n} (A^\dagger)_{k_1}{}^{l_1} \cdots (A^\dagger)_{k_n}{}^{l_n}$ from each term in the results of [Bars and Green 1979], followed by symmetrising all of the i, j pairs, and k, l pairs.

The benefit of the Young projector technique is that the coefficients of each term are easier to determine. We have checked our results against Bars and Green up to $n = 4$.