Fluid dynamics
for the unitary Fermi gas

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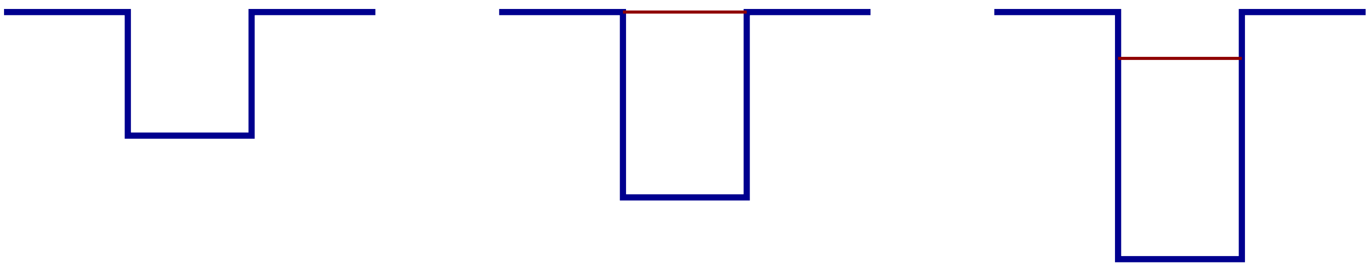
Non-relativistic fermions in unitarity limit

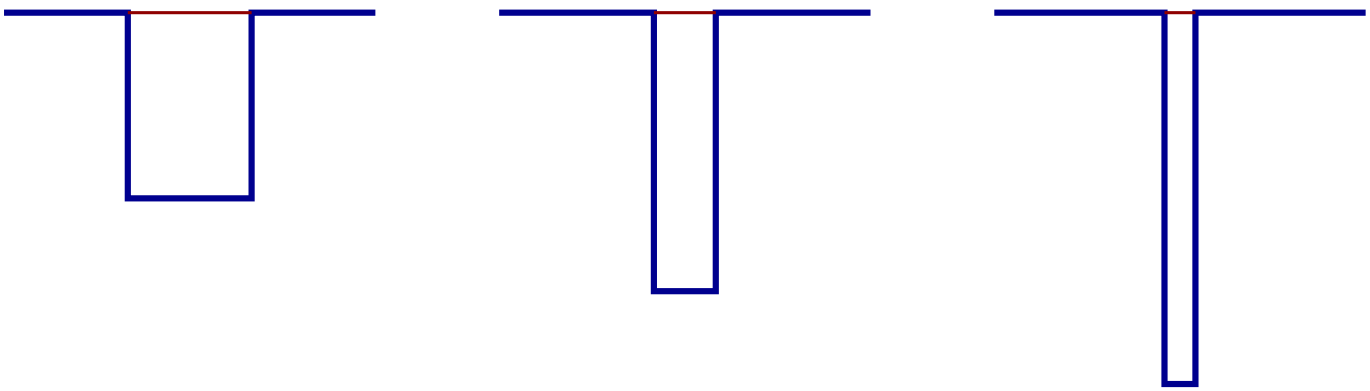
Consider simple square well potential



$a < 0$ $\qquad$ $a = \infty,\ \epsilon_B = 0$ $\qquad$ $a > 0,\ \epsilon_B > 0$

Non-relativistic fermions in unitarity limit

Now take the range to zero, keeping $\epsilon_B \simeq 0$



Universal relations

$$\mathcal{T} = \frac{1}{ik + 1/a} \qquad \epsilon_B = \frac{1}{2ma^2} \qquad \psi_B \sim \frac{1}{\sqrt{a}r}\exp(-r/a)$$

Fermi gas at unitarity: Field Theory

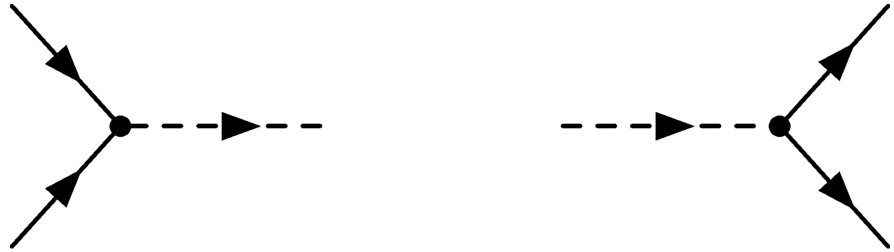
Non-relativistic fermions at low momentum

$$\mathcal{L}_{\text{eff}} = \psi^\dagger \Big(i\partial_0 + \frac{\nabla^2}{2M} \Big) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$

Unitary limit: $a \to \infty$ (DR: $C_0 \to \infty$)

This limit is smooth (HS-trafo, $\Psi = (\psi_\uparrow, \psi_\downarrow^\dagger)$)

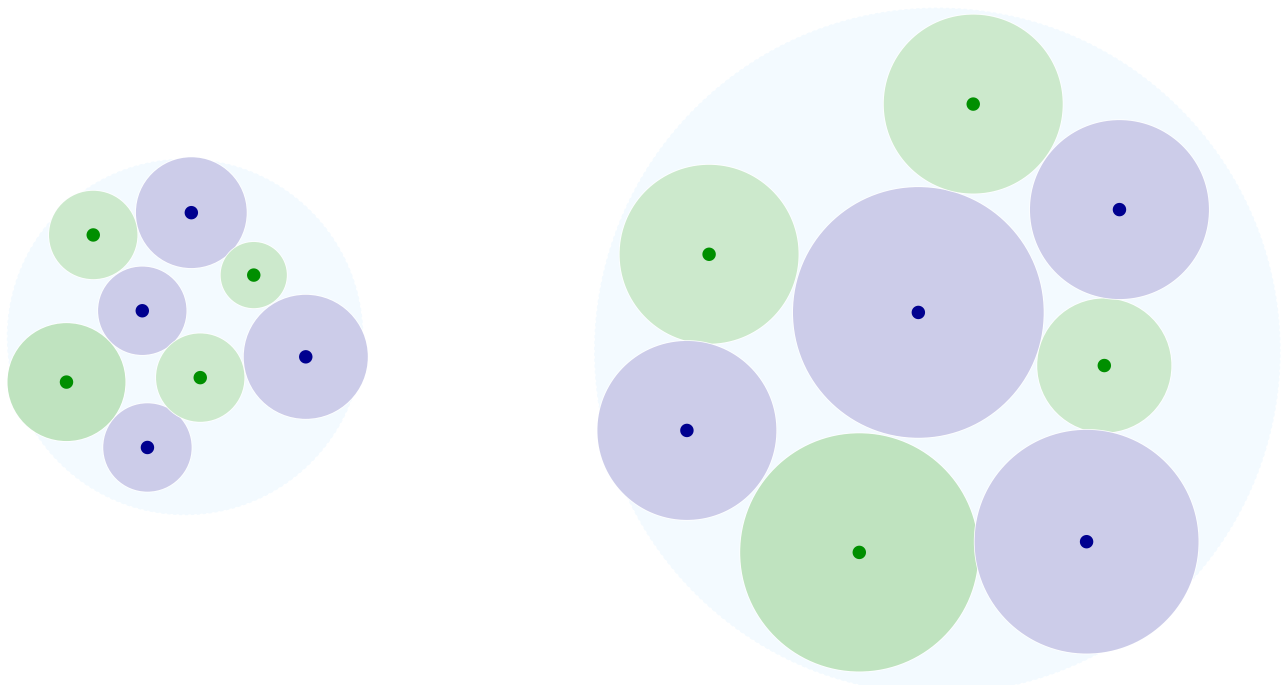
$$\mathcal{L} = \Psi^\dagger \left[i\partial_0 + \sigma_3 \frac{\vec{\nabla}^2}{2m} \right] \Psi + \big(\Psi^\dagger \sigma_+ \Psi \phi + h.c. \big) - \frac{1}{C_0} \phi^* \phi \ ,$$



Scale invariant fluid dynamics

Many body system: Effective cross section $\sigma_{tr} \sim n^{-2/3} \sim (mT_F)^{-1}$

At high temperature $\sigma_{tr} \sim \lambda^2 \sim (mT)^{-1}$



Systems remains hydrodynamic despite expansion

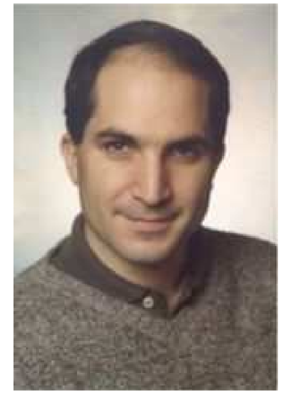
Effective theories for fluids (Unitary Fermi Gas, $T > T_F$)

$$\mathcal{L} = \psi^\dagger \left(i\partial_0 + \frac{\nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi^\dagger \psi)^2$$

$$\frac{\partial f_p}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_p = C[f_p] \qquad \omega < T$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j} \Pi_{ij} = 0 \quad \omega < T \frac{s}{\eta}$$

Effective theories (Strong coupling)



$$\mathcal{L} = \bar{\lambda}(i\sigma \cdot D)\lambda - \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \ldots \Leftrightarrow S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\mathcal{R} + \ldots$$

$$SO(d+2,2) \to Schr(d) \qquad AdS_{d+3} \to Schr^2_d$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}\Pi_{ij} = 0 \quad (\omega < T)$$

Outline

I. Conformal fluid dynamics

II. Kinetic theory

III. Quantum field theory

IV. Holography

V. Experiment

I. Conformal fluid dynamics

Galilean boost	$\vec{x}' = \vec{x} + \vec{v}t$	$t' = t$
Scale trafo	$\vec{x}' = e^s \vec{x}$	$t' = e^{2s} t$
Conformal trafo	$\vec{x}' = \vec{x}/(1+ct)$	$1/t' = 1/t + c$

Ideal fluid dynamics

$$\Pi^0_{ij} = P\delta_{ij} + \rho v_i v_j ,$$

$$P = \frac{2}{3}\mathcal{E}$$

First order viscous hydrodynamics

$$\delta^{(1)}\Pi_{ij} = -\eta\sigma_{ij}, \;\; \sigma_{ij} = \left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}(\nabla \cdot v)\right),$$

$$\zeta = 0$$

Son (2007)

Second order conformal hydrodynamics

Second order gradient corrections to stress tensor

$$\delta^{(2)}\Pi^{ij} = \eta\tau_\pi \left[{}^{\langle}D\sigma^{ij\rangle} + \frac{2}{3}\sigma^{ij}(\nabla\cdot v)\right] + \lambda_1 \sigma^{\langle i}{}_k\sigma^{j\rangle k} + \lambda_2 \sigma^{\langle i}{}_k\Omega^{j\rangle k} + \lambda_3 \Omega^{\langle i}{}_k\Omega^{j\rangle k} + \gamma_1 \nabla^{\langle i}T\nabla^{j\rangle}T + \gamma_2 \nabla^{\langle i}P\nabla^{j\rangle}P + \gamma_3 \nabla^{\langle i}\nabla^{j\rangle}T + \ldots$$

$$D = \partial_0 + v\cdot\nabla \quad A^{\langle ij\rangle} = \frac{1}{2}\left(A_{ij} + A_{ji} - \frac{2}{3}g_{ij}A^k{}_k\right) \quad \Omega^{ij} = \left(\nabla_i v_j - \nabla_j v_i\right)$$

New transport coefficients $\tau_\pi, \lambda_i, \gamma_i$

Can be written as a relaxation equation for $\pi^{ij} \equiv \delta\Pi^{ij}$

$$\pi^{ij} = -\eta\sigma^{ij} - \tau_\pi \left[{}^{\langle}D\pi^{ij\rangle} + \frac{5}{3}(\nabla\cdot v)\pi^{ij}\right] + \ldots$$

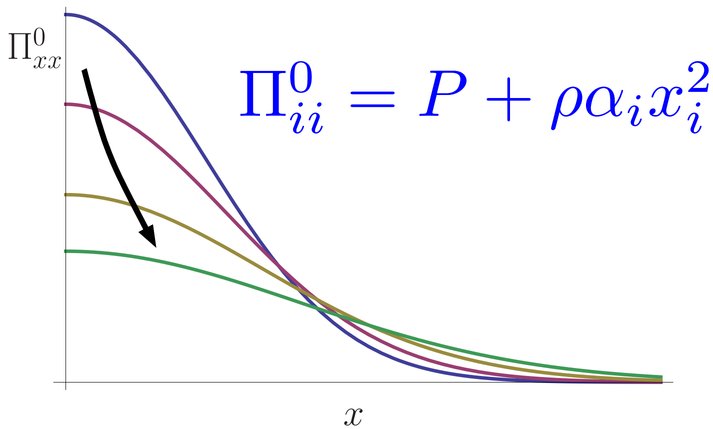
Chao, Schaefer (2011)

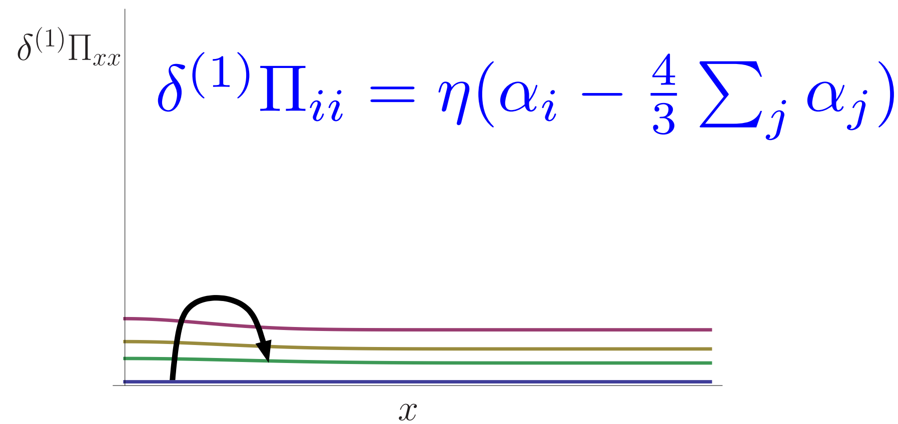
Why second order fluid dynamics?

Consider scaling ("Hubble") expansion

$$\rho(x_i,t) = \rho_0(b_i(t)x_i), \qquad v_i(x_j,t) = \alpha_i(t)x_i, \qquad \alpha_i(t) = \dot{b}_i(t)/b_i(t)$$

Compare ideal and dissipative stresses





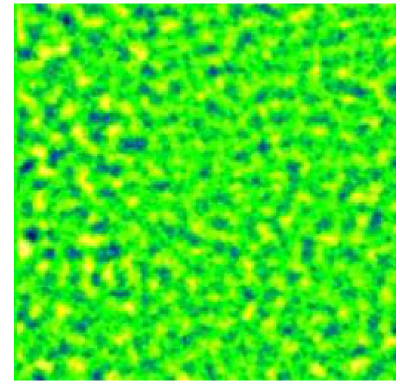
v(ideal stresses) $\sim c_s$ $\qquad$ v(dissipative stresses) $\sim \infty$.

Acausal behavior; hydro always breaks down in the dilute corona.

Solved by relaxation time $\tau_\pi \sim \frac{\eta}{P}$.

Hydrodynamic fluctuations

Hydrodynamic variables fluctuate



$$\langle \delta v_i(x,t)\delta v_j(x',t)\rangle = \frac{T}{\rho}\delta_{ij}\delta(x-x')$$

Linearized hydrodynamics propagates fluctuations as shear or sound

$$\langle \delta v_i^T \delta v_j^T \rangle_{\omega,k} = \frac{2T}{\rho}(\delta_{ij} - \hat{k}_i\hat{k}_j)\frac{\nu k^2}{\omega^2 + (\nu k^2)^2} \qquad \textit{shear}$$

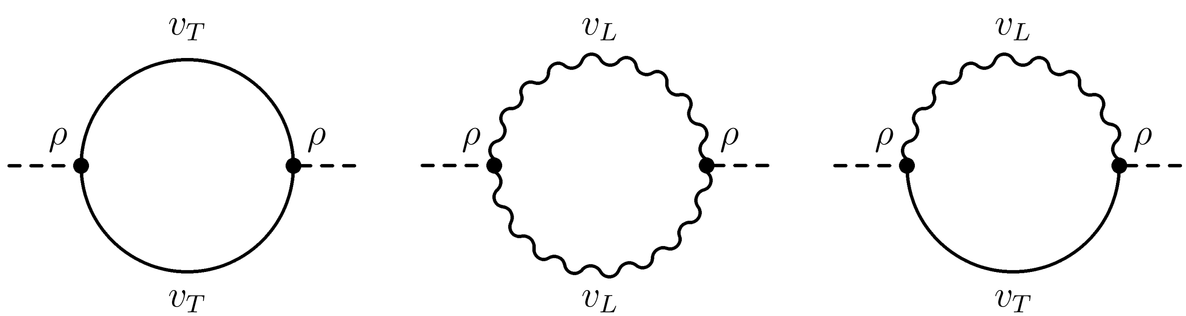
$$\langle \delta v_i^L \delta v_j^L \rangle_{\omega,k} = \frac{2T}{\rho}\hat{k}_i\hat{k}_j\frac{\omega k^2\Gamma}{(\omega^2 - c_s^2 k^2)^2 + (\omega k^2\Gamma)^2} \qquad \textit{sound}$$

$v = v_T + v_L$: $\nabla \cdot v_T = 0$, $\nabla \times v_L = 0$ $\qquad$ $\nu = \eta/\rho$, $\Gamma = \frac{4}{3}\nu + \ldots$

Hydro Loops: "Breakdown" of second order hydro

Correlation function in hydrodynamics

$$G_S^{xyxy} = \langle\{\Pi^{xy}, \Pi^{xy}\}\rangle_{\omega,k} \simeq \rho_0^2 \langle\{v_x v_y, v_x v_y\}\rangle_{\omega,k}$$



Response function, Kubo limit

$$G_R^{xyxy} = P + \delta P - i\omega[\eta + \delta\eta] + \omega^2 \left[\eta\tau_\pi + \delta(\eta\tau_\pi)\right]$$

$$\delta\eta \sim T \left(\frac{\rho}{\eta}\right)^2 \left(\frac{P}{\rho}\right)^{1/2} \qquad \delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}} \left(\frac{\rho}{\eta}\right)^{3/2}$$

Hydro Loops: "Breakdown" of second order hydro

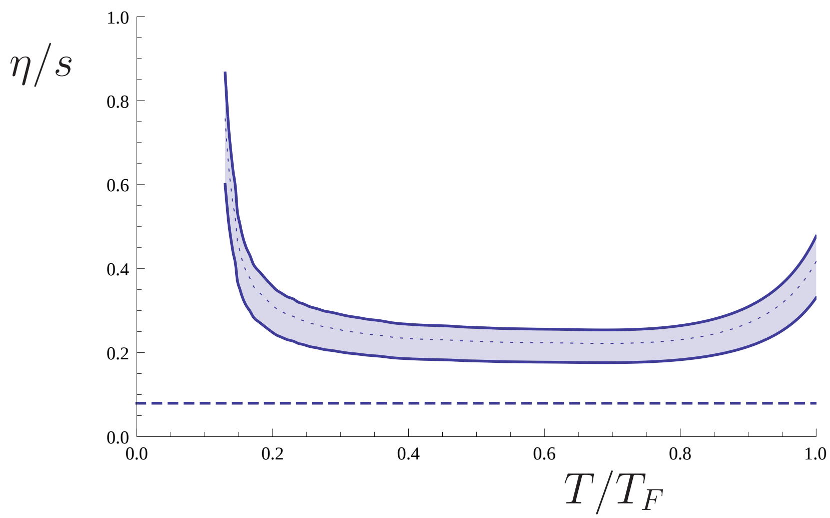
$$\delta\eta \sim T\left(\frac{\rho}{\eta}\right)^2\left(\frac{P}{\rho}\right)^{1/2} \qquad\qquad \delta(\eta\tau_\pi) \sim \frac{1}{\sqrt{\omega}}\left(\frac{\rho}{\eta}\right)^{3/2}$$

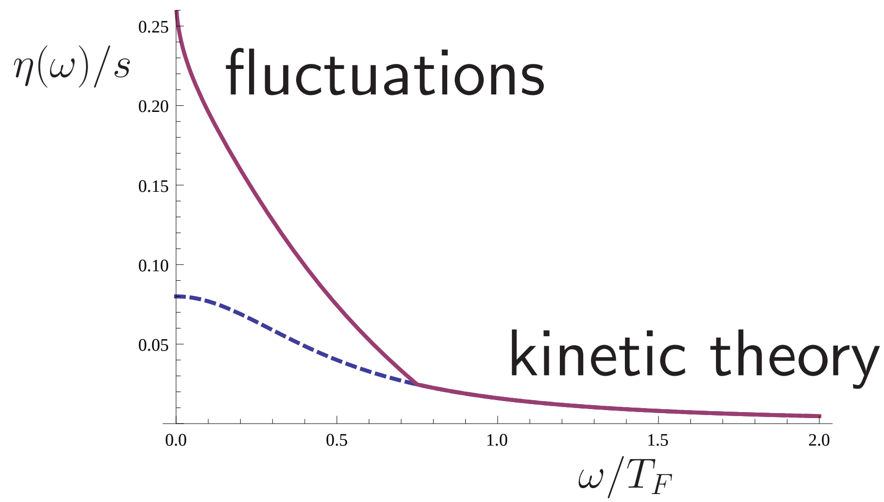
Small shear viscosity enhances fluctuation corrections.

Small η leads to large $\delta\eta$: There must be a bound on η/n.

Relaxation time diverges: 2nd order hydro without fluctuations inconsistent.

Fluctuation induced bound on η/s





$(\eta/s)_{min} \simeq 0.2$

spectral function

non-analytic $\sqrt{\omega}$ term

Schaefer, Chafin (2012), see also Kovtun, Moore, Romatschke (2011)

II. Linear response and kinetic theory

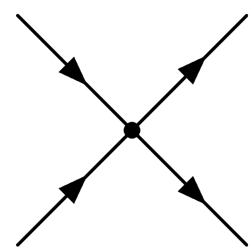
Consider background metric $g_{ij}(t, \mathbf{x}) = \delta_{ij} + h_{ij}(t, \mathbf{x})$. Linear response

$$\delta\Pi^{ij} = -\frac{1}{2} G_R^{ijkl} h_{kl}$$

Kubo relation: $\eta(\omega) = \frac{1}{\omega} \mathrm{Im} G_R^{xyxy}(\omega, 0)$

Kinetic theory: Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{p^i}{m} \frac{\partial}{\partial x^i} - \left(g^{il} \dot{g}_{lj} p^j + \Gamma^i_{jk} \frac{p^j p^k}{m} \right) \frac{\partial}{\partial p^i} \right) f(t, \mathbf{x}, \mathbf{p}) = C[f]$$

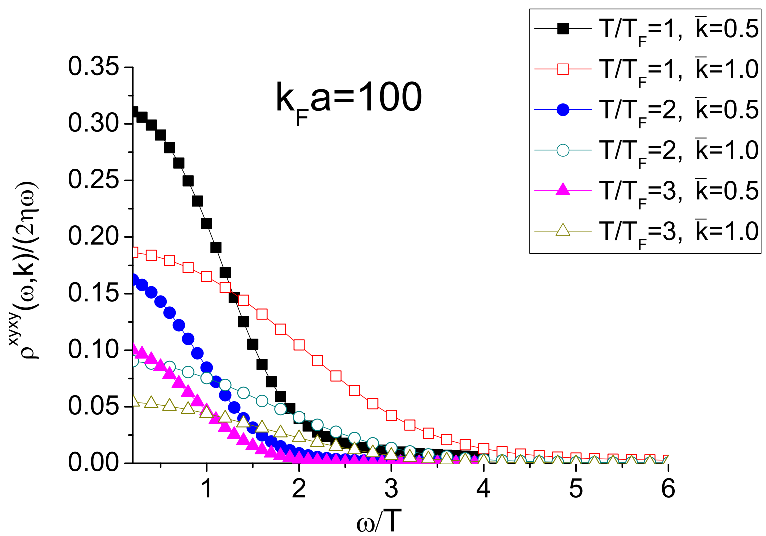
$C[f] =$ 

Kinetic theory

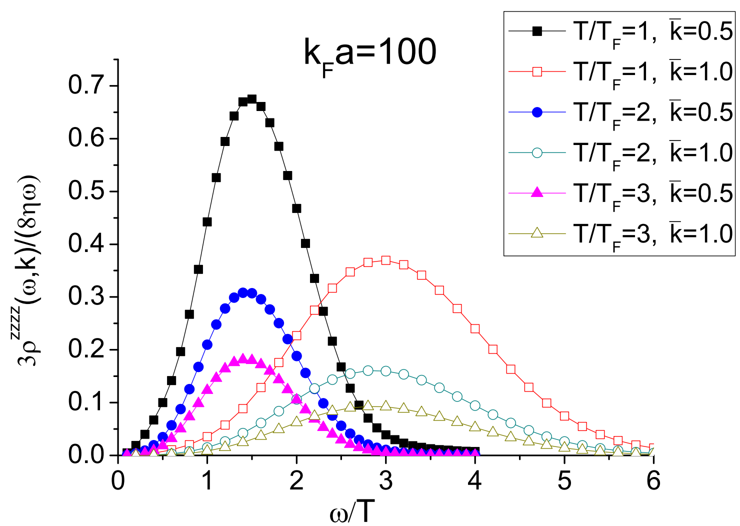
Linearize $f = f_0 + \delta f$, solve for δf, $\hookrightarrow \delta\Pi_{ij}$, $\hookrightarrow G_R$, $\hookrightarrow \eta(\omega)$

$$\eta(\omega) = \frac{\eta}{1+\omega^2\tau_\pi^2} \qquad \eta = \frac{15}{32\sqrt{\pi}}(mT)^{3/2} \qquad \tau_\pi = \frac{\eta}{nT}$$

shear channel



sound channel



Braby, Chao, T.S. (2010)

Second order hydrodynamics from kinetic theory

Chapman-Enskog expansion $f = f_0 + \delta f_1 + \delta f_2 + \ldots$

$\equiv$ Knudsen expansion $\delta f_n = O(Kn^n) = O(\nabla^n)$

$$\begin{aligned}\delta^{(2)}\Pi^{ij} &= \frac{\eta^2}{P}\left[\langle D\sigma^{ij}\rangle + \frac{2}{3}\sigma^{ij}(\nabla\cdot v)\right] \\ &+ \frac{\eta^2}{P}\left[\frac{15}{14}\sigma^{\langle i}{}_k\sigma^{j\rangle k} - \sigma^{\langle i}{}_k\Omega^{j\rangle k}\right] + O(\kappa\eta\nabla^i\nabla^j T)\end{aligned}$$

relaxation time $\quad \tau_\pi = \dfrac{\eta}{P} \simeq \dfrac{\eta}{nT}$

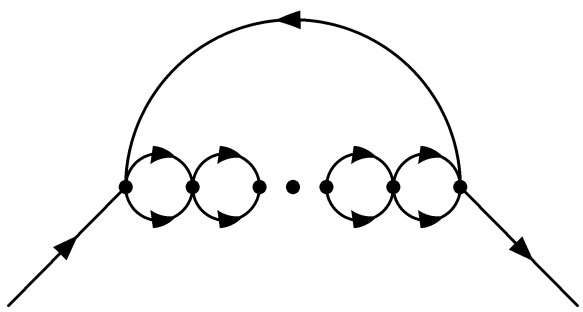
Chao, Schaefer (2012), Schaefer (2014)

Bulk viscosity and conformal symmetry breaking

Conformal symmetry breaking (thermodynamics)

$$1 - \frac{2\mathcal{E}}{3P} = \frac{\langle \mathcal{O}_C \rangle}{12\pi m a P} \sim \frac{1}{6\pi} n\lambda^3 \frac{\lambda}{a}$$

How does this translate into $\zeta \neq 0$? Momentum dependent $m^*(p)$.



$$Im\,\Sigma(k) \sim zT\sqrt{\frac{T}{\epsilon_k}} Erf\left(\sqrt{\frac{\epsilon_k}{T}}\right) \ll T$$

$$Re\,\Sigma(k) \sim zT\frac{\lambda}{a}\sqrt{\frac{T}{\epsilon_k}} F_D\left(\sqrt{\frac{\epsilon_k}{T}}\right)$$

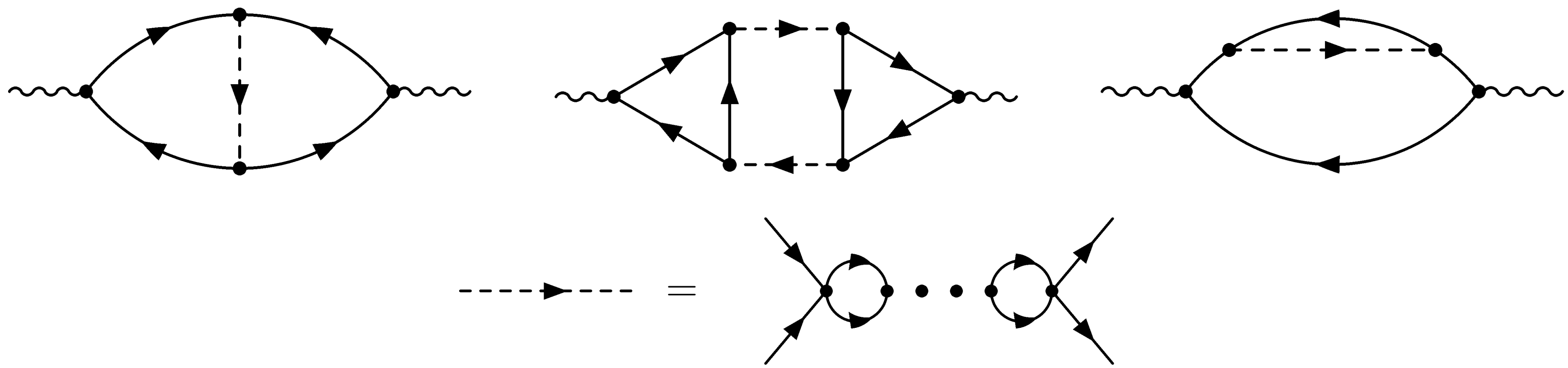
Bulk viscosity

$$\zeta = \frac{1}{24\sqrt{2}\pi}\lambda^{-3}\left(\frac{z\lambda}{a}\right)^2$$

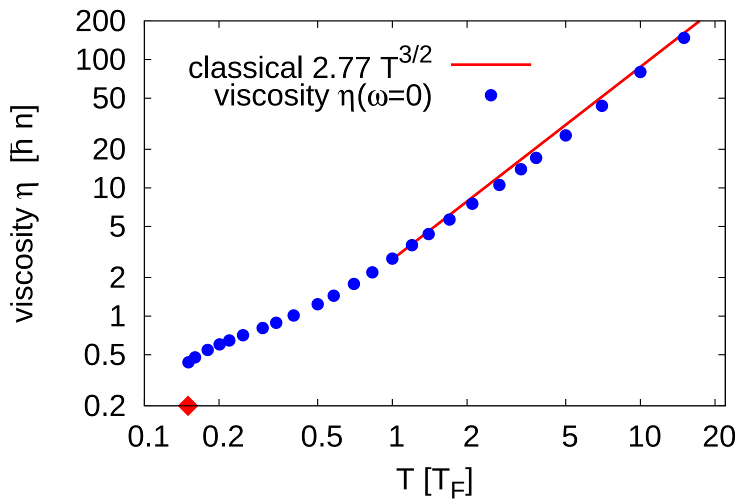
$$\zeta \sim \left(1 - \frac{2\mathcal{E}}{3P}\right)^2 \eta$$

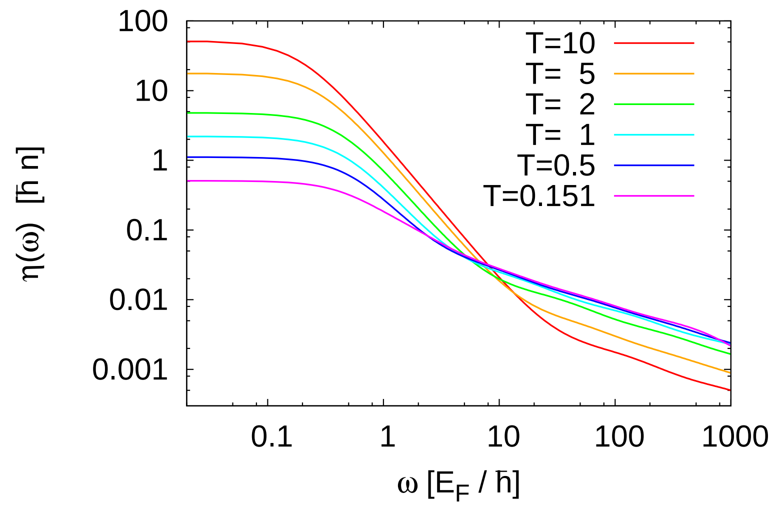
III. Quantum Field Theory

The diagrammatic content of the Boltzmann equation is known: Kubo formula with "Maki-Thompson" + "Azlamov-Larkin" + "Self-energy"



Can be used to extrapolate Boltzmann result to $T \sim T_F$





Enss, Zwerger (2011)

Short time behavior: OPE

Operator product expansion (OPE)

$$\eta(\omega) = \sum_n \frac{\langle \mathcal{O}_n \rangle}{\omega^{(\Delta_n - d)/2}} \qquad \mathcal{O}_n(\lambda^2 t, \lambda x) = \lambda^{-\Delta_n} \mathcal{O}_n(t, x)$$

Leading operator: Contact density (Tan)

$$\mathcal{O}_\mathcal{C} = C_0^2 \psi\psi\psi^\dagger\psi^\dagger = \Phi\Phi^\dagger \qquad \Delta_\mathcal{C} = 4$$

$\eta(\omega) \sim \langle \mathcal{O}_\mathcal{C} \rangle / \sqrt{\omega}$. Asymptotic behavior + analyticity gives sum rule

$$\frac{1}{\pi} \int dw \left[\eta(\omega) - \frac{\langle \mathcal{O}_\mathcal{C} \rangle}{15\pi\sqrt{m\omega}} \right] = \frac{\mathcal{E}}{3}$$

Randeria, Taylor (2010), Enss, Zwerger (2011), Hoffman (2013)

IV. Holography

DLCQ idea: Light cone compactification of relativistic theory in d+2

$$p_\mu p^\mu = 2p_+ p_- - p_\perp^2 = 0 \qquad p_- = \frac{p_\perp^2}{2p_+} \qquad p_+ = \frac{2n+1}{L}$$

Galilean invariant theory in d+1 dimensions.

String theory embedding: Null Melvin Twist

$$AdS_{d+3} \xrightarrow{\text{NMT}} Schr_d^2$$

$$Iso(AdS_{d+3}) = SO(d+2,2) \supset Schr(d)$$

Son (2008), Balasubramanian et al. (2008)

Other ideas: Horava-Lifshitz (Karch, 2013)

Schrödinger Metric

Coordinates $(u, v, \vec{x}, r)$, periodic in v, $\vec{x} = (x, y)$

$$ds^2 = \frac{r^2}{k(r)^{2/3}} \left\{ \left[\frac{1 - f(r)}{4\beta^2} - r^2 f(r) \right] du^2 + \frac{\beta^2 r_+^4}{r^4} dv^2 - [1 + f(r)]\, du\, dv \right\}$$

$$+ k(r)^{1/3} \left\{ r^2\, d\vec{x}^2 + \frac{dr^2}{r^2 f(r)} \right\}$$

Fluctuations $\delta g_x^y = e^{-i\omega u} \chi(\omega, r)$ satisfy $(u = (r_+/r)^2)$

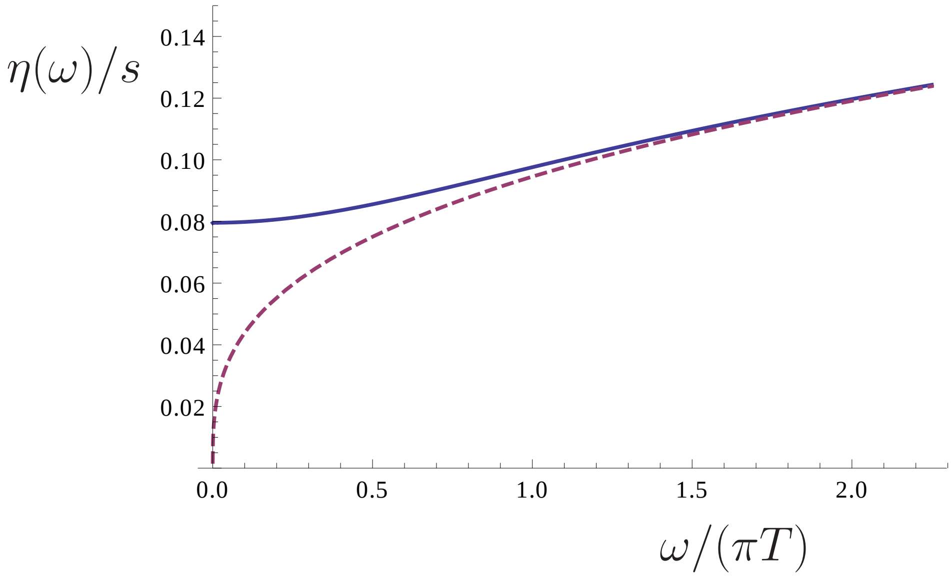
$$\chi''(\omega, u) - \frac{1 + u^2}{f(u) u} \chi'(\omega, u) + \frac{u}{f(u)^2} \mathfrak{w}^2 \chi(\omega, u) = 0$$

Retarded correlation function

$$G_R(\omega) = \frac{\beta r_+^3 \Delta v}{4\pi G_5} \left. \frac{f(u)\chi'(\omega, u)}{u\chi(\omega, u)} \right|_{u \to 0} .$$

Adams et al. (2008), Herzog et al. (2008)

Spectral function



$$\eta(0)/s = 1/(4\pi)$$

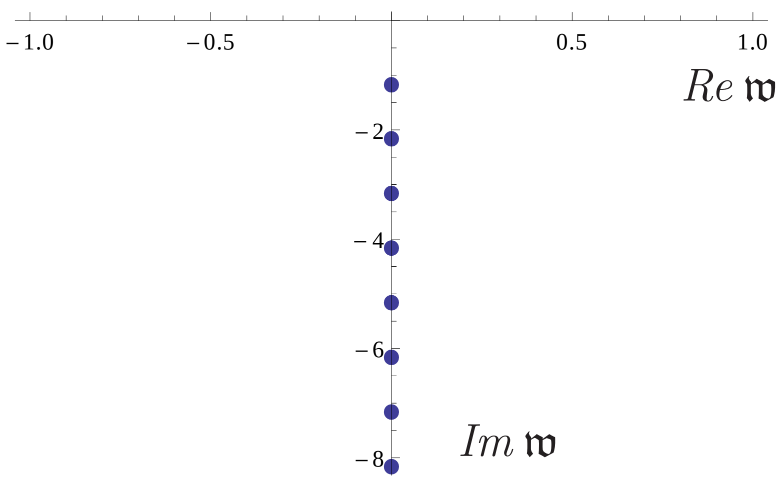
$$\eta(\omega \to \infty) \sim \omega^{1/3}$$

Relaxation time: $G_R(\omega) = P - i\eta\omega + \tau_\pi \eta \omega^2 + \kappa_R k^2$

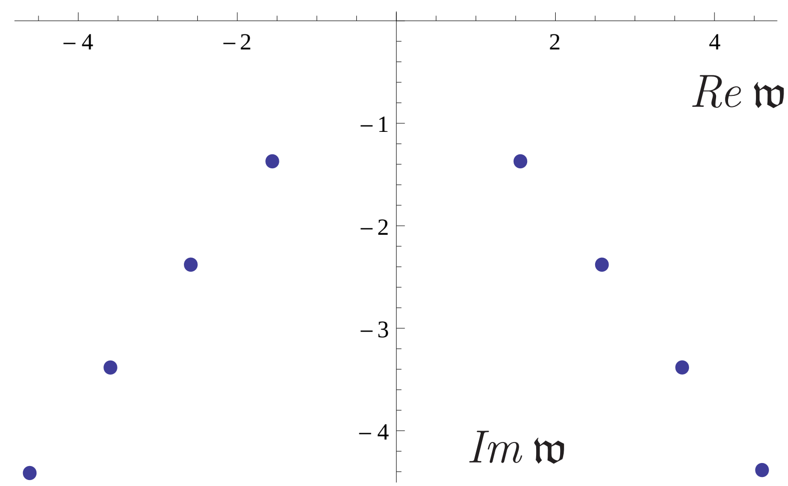
$$\tau_\pi T = -\frac{\log(2)}{2\pi} \qquad AdS_5: \; \tau_\pi T = \frac{2 - \log(2)}{2\pi}$$

Schaefer (2014), BRSSS (2008)

Quasi-normal modes



Sch_2^2

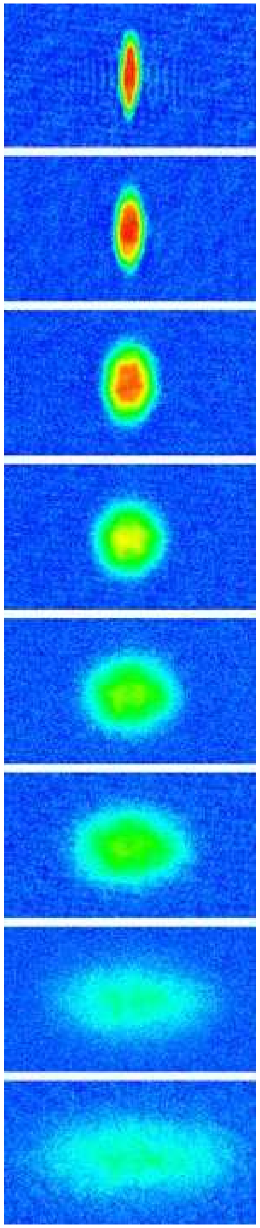


AdS_5

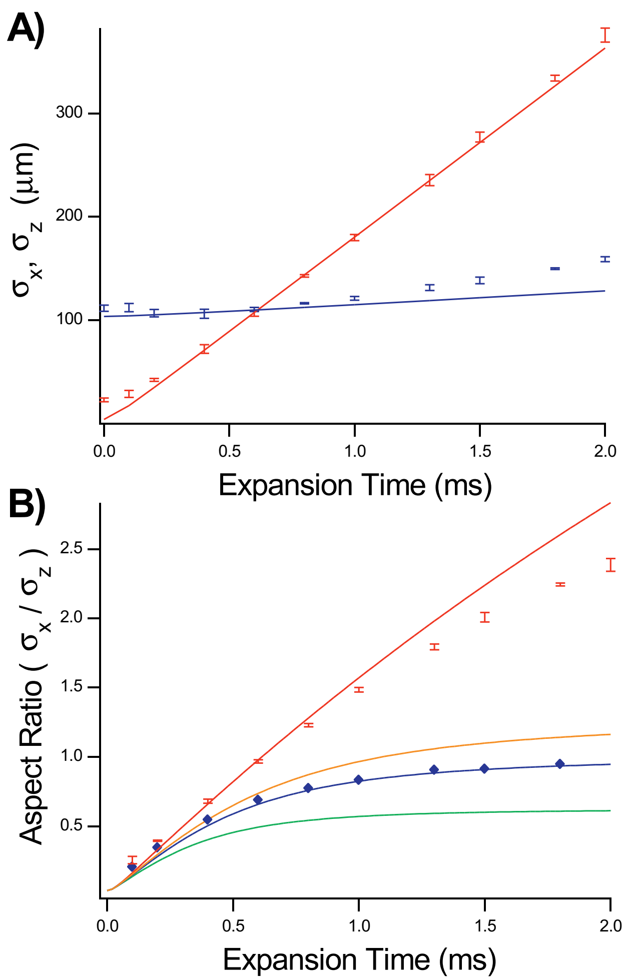
QNM's are stable, $\mathrm{Im}\,\lambda < 0$. Schrödinger metric has unpaired eigenvalues, similar to relaxation time Boltzmann.

Schaefer (2014), Starinets (2002)

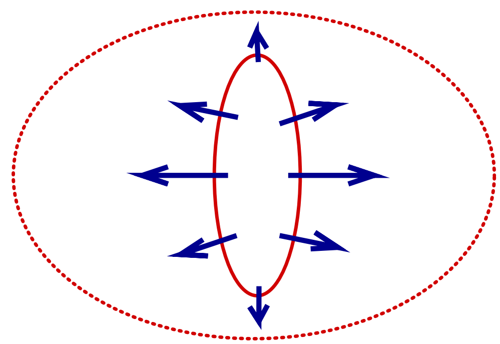
V. Experiments: Flow and Collective Modes



O'Hara et al. (2002)

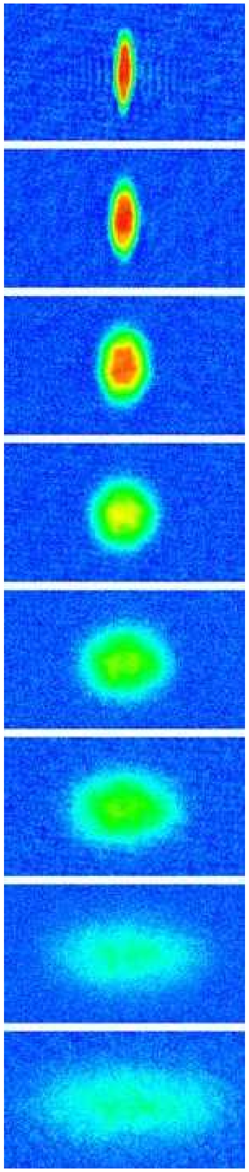


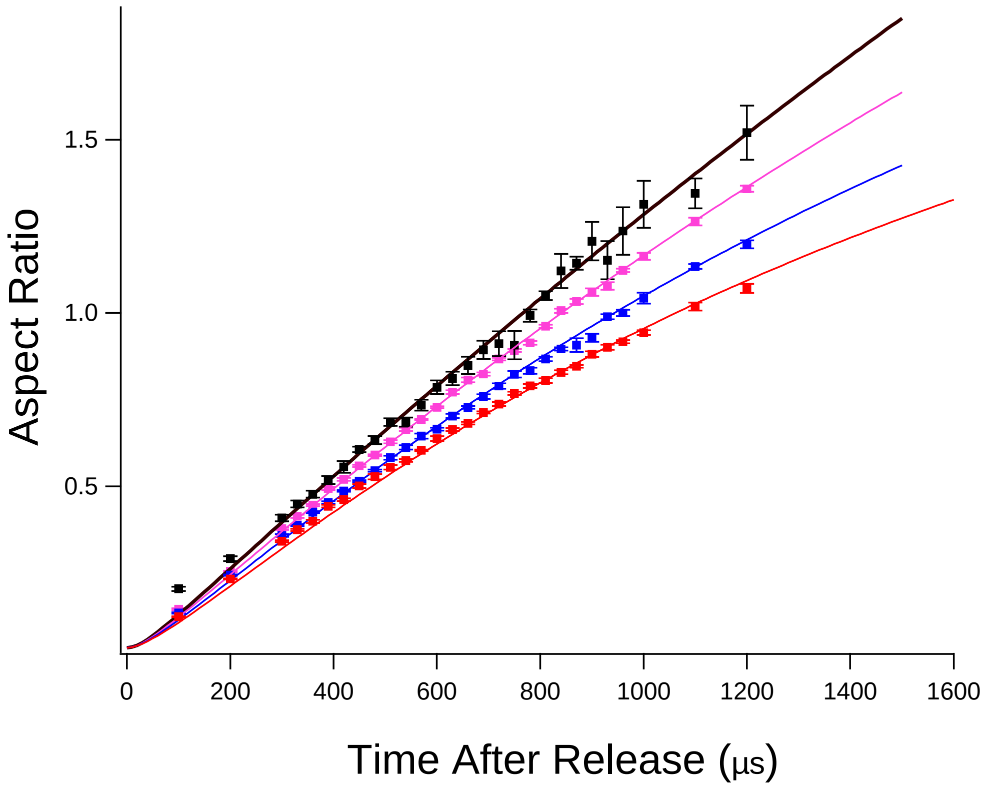
Hydrodynamic expansion converts coordinate space anisotropy to momentum space anisotropy



Elliptic flow: High T limit

Quantum viscosity $\eta = \eta_0 \dfrac{(mT)^{3/2}}{\hbar^2}$





$$\eta = \eta_0 (mT)^{3/2}$$

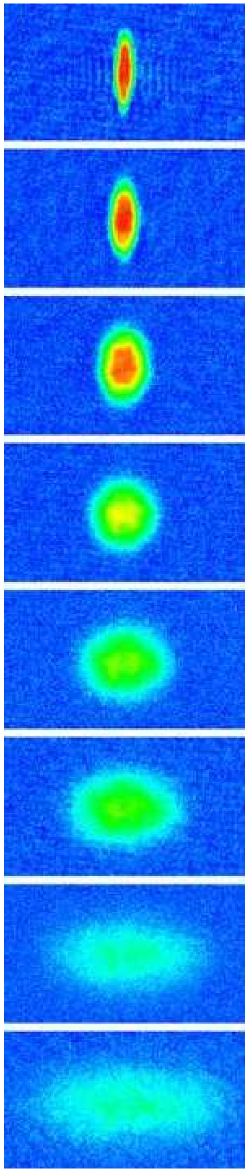
$$\tau_\pi = \eta / P$$

Cao et al., Science (2010)

fit: $\eta_0 = 0.33 \pm 0.04$

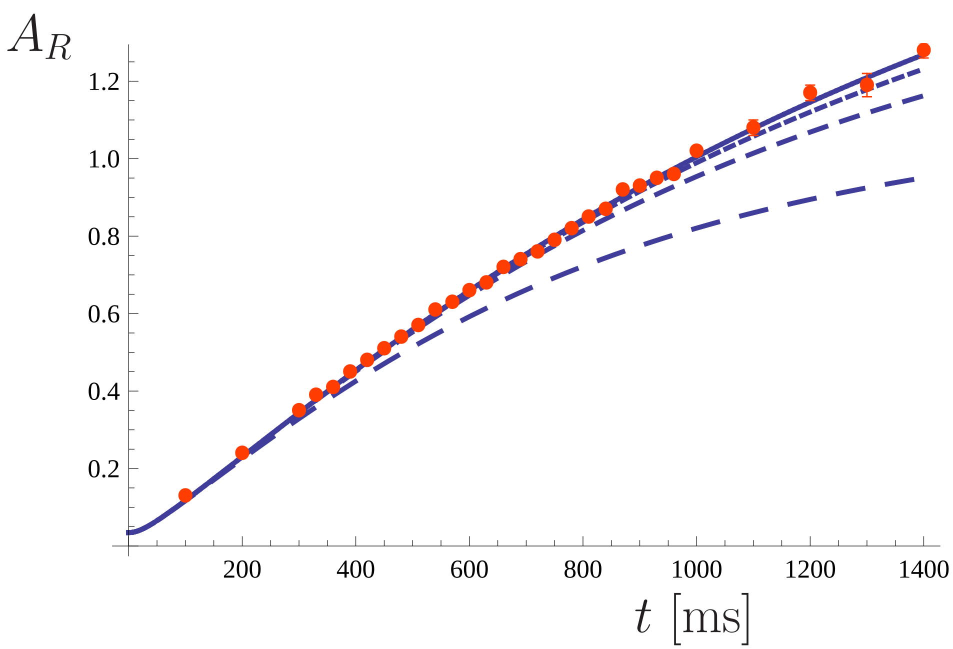
theory: $\eta_0 = \frac{15}{32\sqrt{\pi}} = 0.26$

Elliptic flow: Freezeout?



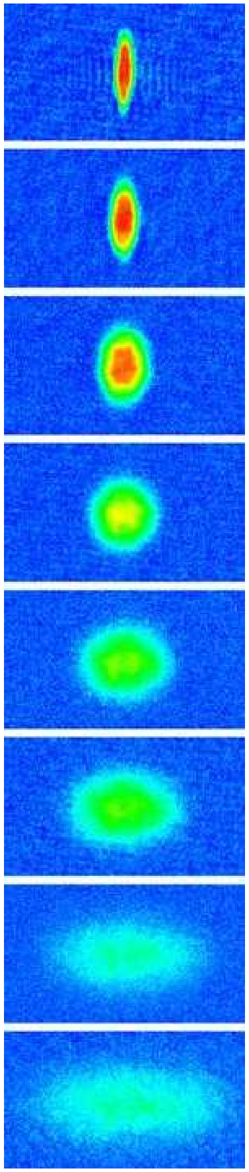
switch from hydro to (weakly collisional) kinetics

at scale factor $b_\perp^{fr} = 1, 5, 10, 20$

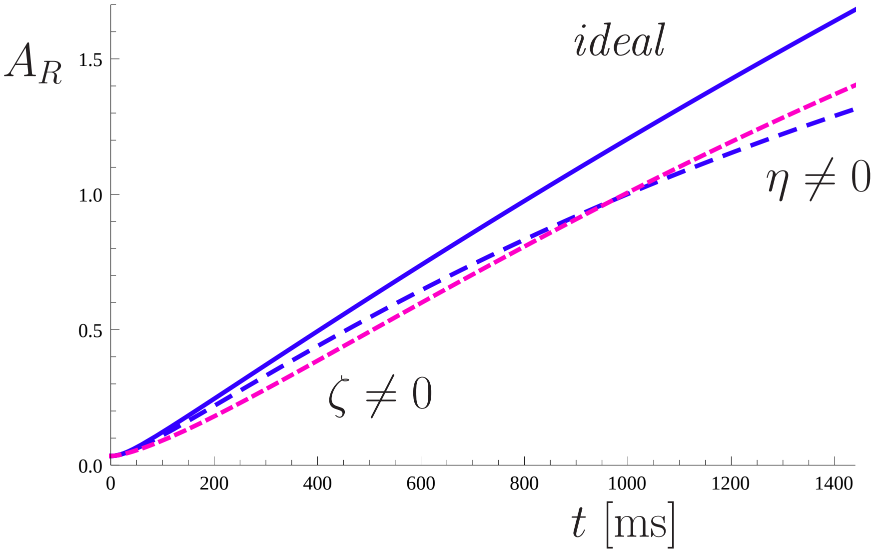


no freezeout seen in the data

Elliptic flow: Shear vs bulk viscosity



Dissipative hydro with both η, ζ



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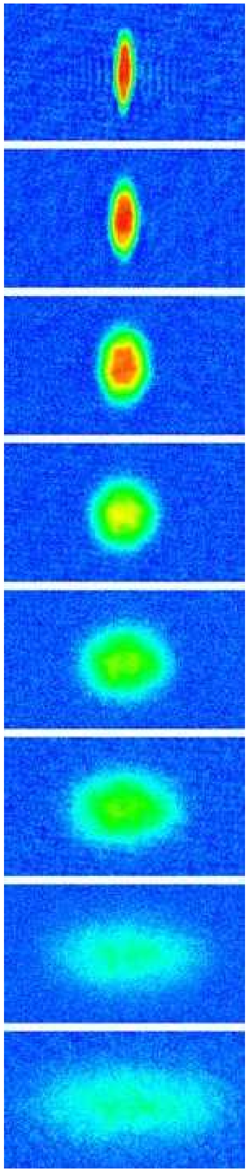
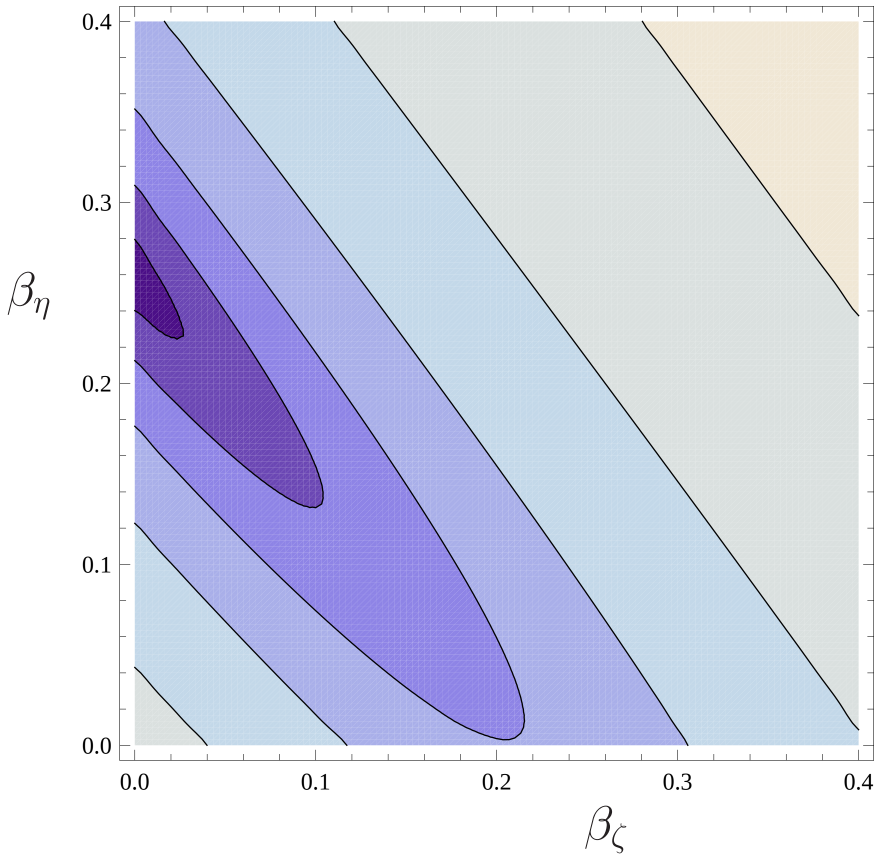


Dissipative hydro with both η, ζ

$$\beta_{\eta,\zeta} = \frac{[\eta,\zeta]}{n}\,\frac{E_F}{E}\,\frac{1}{(3\lambda N)^{1/3}}$$



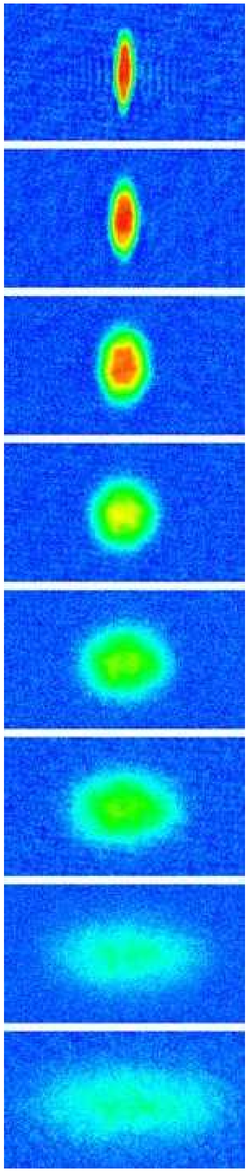
$\eta \gg \zeta$

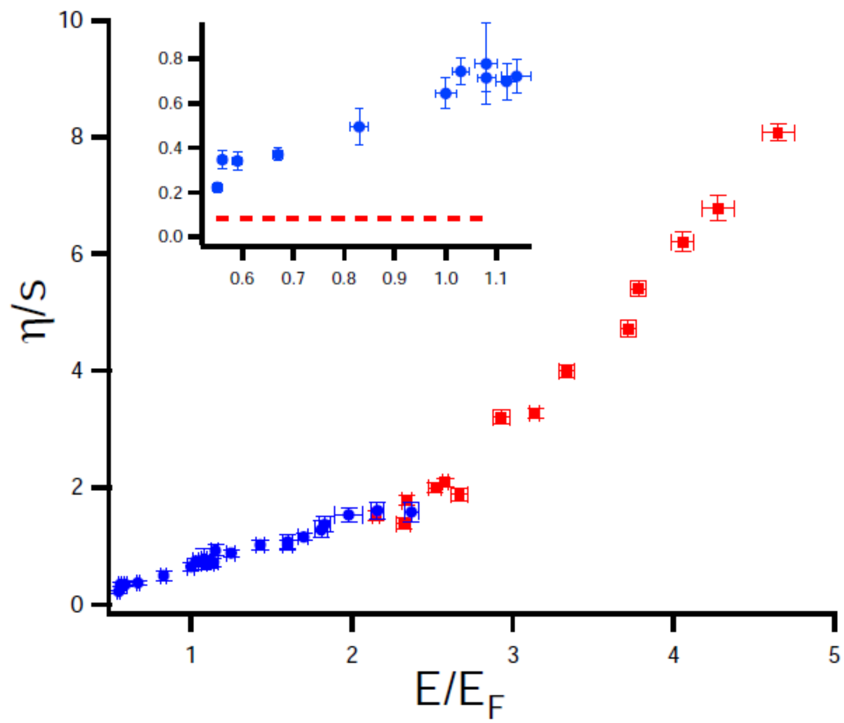
Dusling, Schaefer (2010)

Viscosity to entropy density ratio

consider both collective modes (low T)

and elliptic flow (high T)





Cao et al., Science (2010)

$$\eta/s \leq 0.4$$

Lessons & Outlook

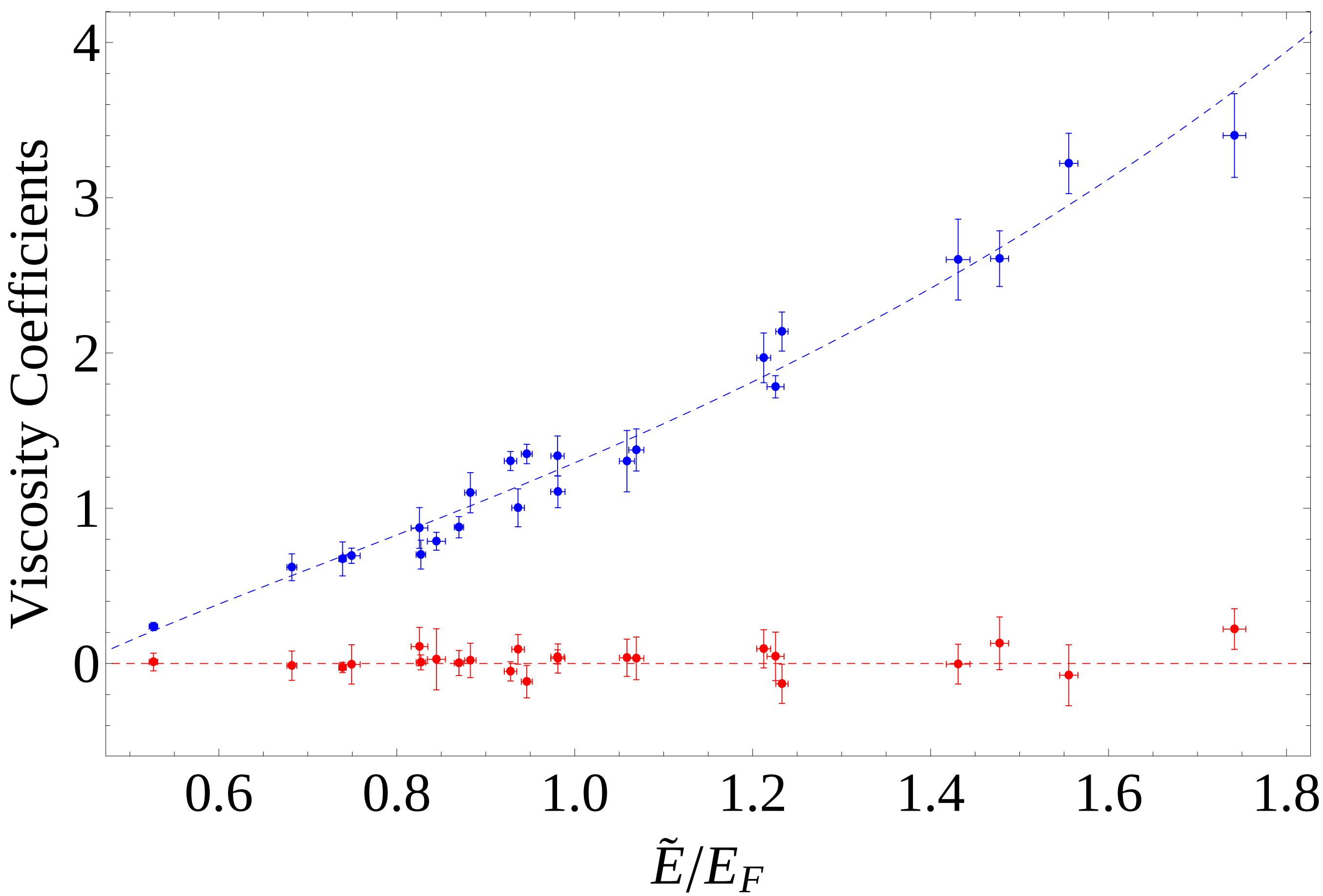
Experiment: Main issue is temperature, density dependence of η/s. How to unfold?

Need hydro codes that exit “gracefully” (LBE, anisotropic hydro, hydro+cascade)

Quasi-particles vs quasi-normal modes (kinetics vs holography) unresolved. Need better holographic models, improved lattice calculations.

Can we observe breaking of scale invariance and the return of bulk viscosity away from unitarity? Can we measure η and ζ_3 in the superfluid phase?

Elliptic flow: Shear vs bulk viscosity



Elliptic flow away from unitarity