Towards understanding chiral vortical effect

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Consider a hydrodynamic theory with $u^\mu$, $T$, and $\mu$

The motion of the medium is governed by

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu j^\mu = 0$$

Specifically:

$$j^\mu = nu^\mu - 6T(\gamma^\mu + u^\nu u^\mu) \partial_\nu (\frac{\mu}{T}) + \varepsilon \omega^\mu$$

Anomaly
Here vorticity is defined by

\[ \omega^\mu = \frac{1}{2} \varepsilon^{\mu \nu \lambda \sigma} \omega_{\nu \lambda} \omega_{\sigma} \]

\[ \xi : \text{D. Son and P. Sarówka (2009)} \]

But vorticity in flat space is problematic

for relativistic theories.
we know this problem from angular momentum...

Lorentz group is origin dependent.
Example: Consider fluids motion confined to a plane 2D: $\omega_z = \frac{1}{r} \frac{\partial}{\partial r} (ru_\phi)$

$u_\phi = ar^n$

$\Rightarrow \omega_z = a(r+1)n^{-1} \quad n > 1$

Vorticity becomes larger as we go far from the origin => fluid's speed $> c$. 
Solution: curved space

We already know that curved-space configurations allow vorticity.

\[ T^\mu_\nu = T^\mu_\nu_{\text{ideal}} + \ldots + \lambda_1 \, g^\mu_\nu \, \mathcal{E} + \lambda_2 \, g^\mu_\nu \, \Omega \\
\mathcal{E}^\nu_\mu = \nabla^\nu u - \nabla u + \lambda_3 \, g^\mu_\nu \, \mathcal{E} \]
Consider small perturbations around flat space:

$$\vartheta_{\mu \nu}(x) = \eta_{\mu \nu} + h_{\mu \nu}(x)$$

with $h_{\mu \nu}(x) \neq 0 \quad (t, x, y, z)$

rest frame: $\nu_{\mu} = (-1, h_{0x}, 0, 0)$

You get vorticity vector $\omega = -\frac{1}{2} \partial_y h_{0x}$
Our goal is to understand the chiral vortical effect microscopically.

Let's study fermions on curved space with a chemical potential as a warm-up.

Local Lorentz frames: \( e_\alpha \mu e_\beta^\mu = \gamma_{\alpha \beta} \) \( e_\alpha \mu e^\alpha_\nu = g_{\mu \nu} \)
Modifications:

\[ \gamma^\mu = \gamma_\alpha \epsilon^{\alpha \mu} \]

\[ \left\{ \gamma_\alpha, \gamma_\beta \right\} = 2 \eta_{\alpha \beta} \]

\[ \left\{ \gamma^\mu, \gamma^\nu \right\} = 2 g^{\mu \nu} \]

Lagrangian:

\[ L = i g^\frac{1}{2} \bar{\psi} \sigma^\mu \left( \partial_\mu + \frac{1}{2} G_{[\alpha \beta]} \omega^{\alpha \beta}_\mu \right) \]

\[ G_{[\alpha \beta]} \sim [\gamma_\alpha, \gamma_\beta] \]

\[ \omega^{\alpha \beta}_\mu = e^\alpha e^{\beta \gamma} \Gamma^{\gamma \mu \nu} - e^{[\alpha} e_{\beta]} \mu e^{\beta \nu} \]
We can calculate the Hamiltonian: $H = \mathcal{H} - L$

or even better $\hat{H}^2$

$$\hat{H}^2 = \left( -i \gamma^i \frac{\partial}{\partial x_i} - i \gamma^\mu \frac{\partial}{\partial x^\mu} G_{\alpha\beta} \omega^{\alpha\beta} + i \mu \theta^i \right) \overleftarrow{\partial^i}$$

$$H^2 = \overline{\psi} \left[ \overrightarrow{p^2} + \frac{1}{2} \mu \beta \omega^2 \right] \psi$$
we can solve the equation of motion:

\[
\omega \chi + \left( -\omega - \mu \right) \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \chi - \omega^2 \left( -\frac{\omega_x^2}{2} + \partial_x \right) - \frac{\partial^2}{\partial z^2} \chi = 0
\]

\[
\chi = \begin{pmatrix}
-\partial_y + \sqrt{\partial^2 + (\partial_x + i \omega_z^2)^2} \\
i \partial_z - \frac{\omega_z^2}{2} - \partial_x
\end{pmatrix}
\begin{pmatrix}
-\partial_y + \sqrt{\partial^2 + (\partial_x + i \omega_z^2)^2} \\
i \partial_z - \frac{\omega_z^2}{2} - \partial_x
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i \partial_z - \frac{\omega_z^2}{2} - \partial_x
\end{pmatrix}
\]
\[ \begin{aligned}
\omega^+ - \mu &= -\sqrt{\mathcal{D}_1^2 + \left(\mathcal{D}_2 + i\omega_2^2\right)^2} \\
\omega^- - \mu &= +\sqrt{\mathcal{D}_1^2 + \left(\mathcal{D}_2 + i\omega_2^2\right)^2}
\end{aligned} \]

Poincaré generators:

\[ M^{\mu \rho} = \int_\Sigma d\Sigma^\alpha \left( x^\alpha T^{\rho \beta} - x^\beta T^{\rho \alpha} \right) \]

with

\[ T = \frac{i\sqrt{8}}{\mathcal{D}_1} \left( \nabla \bar{\psi} \gamma^\mu \psi + \nabla^2 \bar{\psi} \gamma^\mu \psi - \bar{\psi} \gamma^\mu \nabla \psi - \bar{\psi} \gamma^\mu \nabla \psi \right) \]
Conclusion:

- It's possible to study a theory microscopically with vorticity using the curved space methods.
- We can apply it to study chiral vortical effect by computing axial current.