Chiral Kinetic Theory

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What is Chiral Magnetic Effect?

$$\boldsymbol{J} \sim \boldsymbol{B}$$

What is special about $\boldsymbol{J} \sim \boldsymbol{B}$? (compare to $\boldsymbol{J} \sim \boldsymbol{E}$)

- Parity has to be broken (before $\boldsymbol{B}$)
- $\boldsymbol{J}$ cannot be dissipative, since $\boldsymbol{B}$ cannot do work.

Chiral Magnetic Effect and Anomaly

Consider free right-handed fermion.

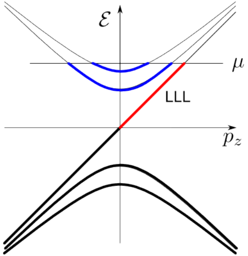
- Lowest Landau level is chiral

$$\boldsymbol{J} = \frac{1}{4\pi^2}\mu \boldsymbol{B}$$

- Nielsen-Ninomia (1983):

$$\boldsymbol{E}\cdot\boldsymbol{J} = \mu\frac{\boldsymbol{E}\cdot\boldsymbol{B}}{4\pi^2} = \mu\frac{dn}{dt}$$

work = energy change



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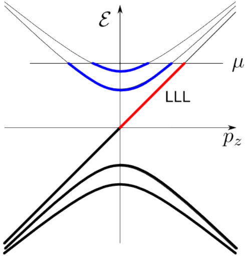

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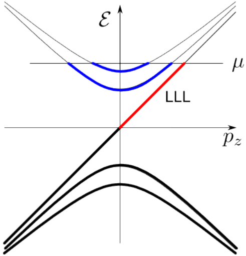

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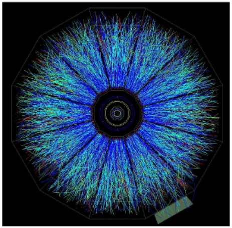
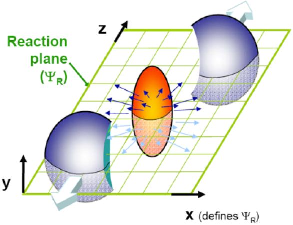
CME in Heavy-Ion Collisions

Kharzeev-McLerran-Warringa (2007):

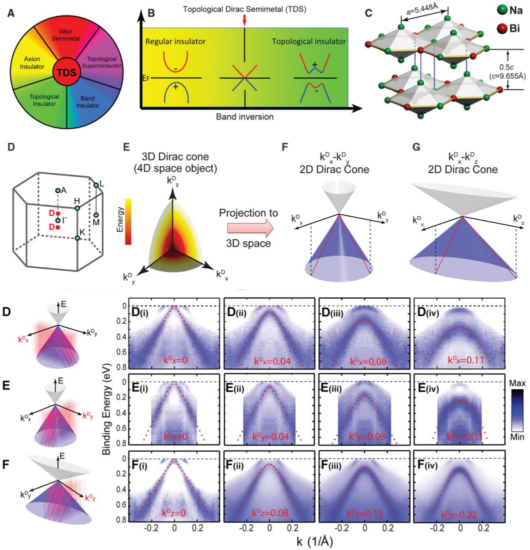
Strong magnetic field
+
topological / net chirality fluctuations
=
fluctuations of charge asymmetry wrt reaction plane.





$$\boldsymbol{J} \sim (N_R - N_L)\boldsymbol{B}$$

"3D graphene"



Liu *et al*, Science 343(2014)864

CME and CVE in Relativistic Hydrodynamics

- Chiral Vortical Effect: $\boldsymbol{J} \sim \boldsymbol{\omega}$

  Vilenkin (1980) – ν emmision from rotating star / BH.

  Rediscovered in AdS/CFT by Erdmenger *et al* (2009)

- Son-Surowka (2009): hydrodynamics of anomalous current

$$\partial_\alpha T^{\alpha\beta} = F^{\beta\gamma} J_\gamma; \qquad \partial_\alpha J^\alpha = C \boldsymbol{E} \cdot \boldsymbol{B}$$

  requires constitutive equation:

$$\boldsymbol{J} = \text{diffusion} + C\mu\boldsymbol{B} + C(\mu^2 + \mathcal{O}(T^2))\boldsymbol{\omega}$$

  • Nondissipative • Same C as anomaly • Finite at $T = 0$

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Motivation

- Interesting applications of CME/CVE in non-equilibrium conditions – such as heavy-ion collisions.
- Kinetic theory: non-equilibrium description of CME/CVE
- Important for microscopic understanding of CME/CVE.
- Condensed matter literature (canonical quantization, massive Dirac eq.):

  Sundaram-Niu (1999), ...

  Field theory (near equilibrium):

  Son-Yamamoto (2012), Manuel-Torres-Rincon (2014)

  Path integral quantization for Weyl particle: MS-Yin (2012)

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Outline

- Introduction: CME, CVE, kinetic theory
- Path integral for Weyl particle, classical limit, Berry monopole, CME and anomaly

  With Yi Yin, PRL 109(2012)162001

- Lorentz invariance in CKT:
  - Modified boost (side-shift), physical meaning
  - Magnetization current and CVE in kinetic theory

  With J. Chen, D. Son; Y. Yin, H.-U. Yee, arXiv:1404.5963

The Kinetic Approach and the Anomaly

- Kin. regime: collisions are rare enough that motion is classical.

Each particle follows classical trajectory $\boldsymbol{x}(t)$, $\boldsymbol{p}(t)$. A “cloud” $f(\boldsymbol{x},\boldsymbol{p})$ evolves with time. In a comoving 6-volume, the number of particles can only be changed by collisions:

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} + \frac{\partial f}{\partial \boldsymbol{p}}\dot{\boldsymbol{p}} = C[f].$$

Ignore collisions for now.

- The number of particles in the phase space *cannot change*?
- How can *classical* equation account for *quantum* anomaly?

Berry monopole

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Equations of motion from path integral

Take Weyl Hamiltonian: $\mathcal{H} = \boldsymbol{\sigma} \cdot (\boldsymbol{p} - \boldsymbol{A}) + \Phi$

Feynman path integral for an amplitude:

$$\mathcal{A}_{fi} = \left[\int \mathcal{D}[\boldsymbol{x}, \boldsymbol{p}] \mathcal{P} \exp \left\{ i \int_{t_i}^{t_f} \boldsymbol{p} \cdot d\boldsymbol{x} - \mathcal{H}\, dt \right\} \right]_{fi}$$

Contains path-ordered 2x2 matrix product of $\exp\{-i\boldsymbol{\sigma} \cdot \boldsymbol{p}(t)\Delta t\}$.

In the classical limit can be diagonized (to $\mathcal{O}(\hbar)$): $V_{\boldsymbol{p}}^{\dagger}\, \boldsymbol{\sigma} \cdot \boldsymbol{p}\, V_{\boldsymbol{p}} = |\boldsymbol{p}|\sigma_3$

$$\mathcal{A}_{fi} \sim \int \mathcal{D}[\boldsymbol{x}, \boldsymbol{p}] \exp \{ i\mathcal{I} \}$$

PRL 109(2012)162001, arXiv:1404.5963

Action

$$\mathcal{I} = \int (\boldsymbol{p} + \boldsymbol{A}) \cdot d\boldsymbol{x} - (\mathcal{E} + \Phi) dt \underbrace{- \boldsymbol{a_p} \cdot d\boldsymbol{p}}_{\text{Berry phase}}$$

Equations of motion

$$\dot{\boldsymbol{x}} - \frac{\partial \mathcal{E}}{\partial \boldsymbol{p}} - \dot{\boldsymbol{p}} \times \boldsymbol{b} = 0;$$

$$\dot{\boldsymbol{p}} - \boldsymbol{E} - \dot{\boldsymbol{x}} \times \boldsymbol{B} = 0;$$

with $\mathcal{E} \equiv |\boldsymbol{p}| - \dfrac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|}$.

"Abelian projection":

$$\boldsymbol{a_p} = [-iV_{\boldsymbol{p}}^{\dagger} \boldsymbol{\nabla}_{\boldsymbol{p}} V_{\boldsymbol{p}}]_{++}$$

Berry curvature:

$$\boldsymbol{b} \equiv \boldsymbol{\nabla}_{\boldsymbol{p}} \times \boldsymbol{a_p} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}.$$

NB: the invariant measure on the phase space is $\dfrac{d^3\boldsymbol{x}\, d^3\boldsymbol{p}}{(2\pi)^3} \sqrt{G}$,

where $G = (1 + \boldsymbol{b} \cdot \boldsymbol{B})^2$ is the det of the 6x6 matrix of $\dot{\boldsymbol{x}}$, $\dot{\boldsymbol{p}}$ coeffs.

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Chiral anomaly

Liouville equation is violated at $\boldsymbol{p}=0$:

$$\frac{\partial}{\partial t}\sqrt{G}+\frac{\partial}{\partial \boldsymbol{x}}(\sqrt{G}\dot{\boldsymbol{x}})+\frac{\partial}{\partial \boldsymbol{p}}(\sqrt{G}\dot{\boldsymbol{p}})=(\boldsymbol{E}\cdot\boldsymbol{B})\,\underbrace{(\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot\boldsymbol{b})}_{2\pi\delta^3(\boldsymbol{p})},$$

Integrate kinetic equation

$$\int_{\boldsymbol{p}}\sqrt{G}\Big(\quad \partial_t f+\dot{\boldsymbol{x}}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}f+\dot{\boldsymbol{p}}\cdot\boldsymbol{\nabla}_{\boldsymbol{p}}f=0\quad\Big),$$

and find for the space-time current $\boldsymbol{j}=\int_{\boldsymbol{p}}\sqrt{G}f\dot{\boldsymbol{x}},\quad n=\int_{\boldsymbol{p}}\sqrt{G}f$

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}=\frac{1}{4\pi^2}\boldsymbol{E}\cdot\boldsymbol{B}\,f|_{\boldsymbol{p}=0}\,,$$

Berry “monopole” at $p=0$ acts as source/sink of particle number current.

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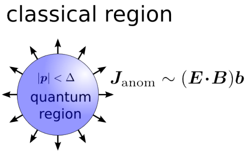

CME

Using eoms:

$$\boldsymbol{j} = \int_{\boldsymbol{p}} \sqrt{G} f \, \dot{\boldsymbol{x}} = \underbrace{\underbrace{\int_{\boldsymbol{p}} f \frac{\partial \mathcal{E}}{\partial \boldsymbol{p}}}_{\text{normal current}} + \underbrace{\boldsymbol{E} \times \int_{\boldsymbol{p}} f \boldsymbol{b}}_{\text{anom. Hall current}}}_{\text{=0 in equilibrium}} + \underbrace{\boldsymbol{B} \int_{\boldsymbol{p}} f (\hat{\boldsymbol{p}} \cdot \boldsymbol{b})}_{\text{CME}}$$

Lorentz invariance?

$$\mathcal{I} = \int (\boldsymbol{p} + \boldsymbol{A}) \cdot d\boldsymbol{x} - (|\boldsymbol{p}| + \Phi) dt - \boldsymbol{a_p} \cdot d\boldsymbol{p} + \frac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|} dt$$

The magnetic moment ($\boldsymbol{\mu} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}$) coupling is essential for Lorentz invariance at order $\mathcal{O}(\hbar)$.

Modified Lorentz transfromation:

$$\delta \boldsymbol{x} = \boldsymbol{\beta} t + \frac{\boldsymbol{\beta} \times \hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}, \quad \delta \boldsymbol{p} = \boldsymbol{\beta} \mathcal{E} + \frac{\boldsymbol{\beta} \times \hat{\boldsymbol{p}}}{2|\boldsymbol{p}|} \times \boldsymbol{B}, \quad \delta t = \boldsymbol{\beta} \cdot \boldsymbol{x}.$$

arXiv:1404.5963

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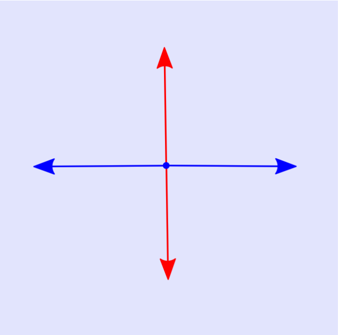
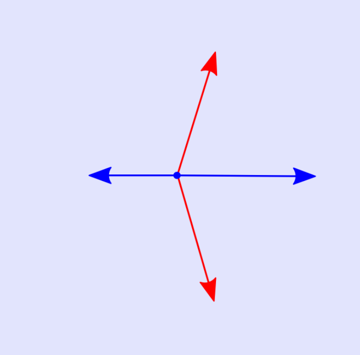
Boost, shift and angular momentum conservation



+ boost =

$\longrightarrow \beta$



$$\boldsymbol{P}_{\text{in}} = \boldsymbol{P}_{\text{out}} = 0$$
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$$\boldsymbol{L}_{\text{in}} = 0, \quad \boldsymbol{L}_{\text{out}} = 0???$$

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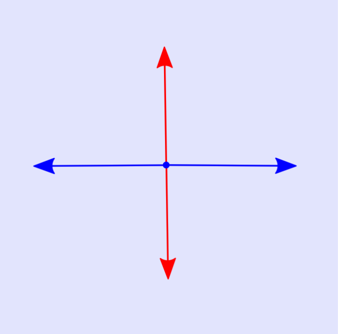

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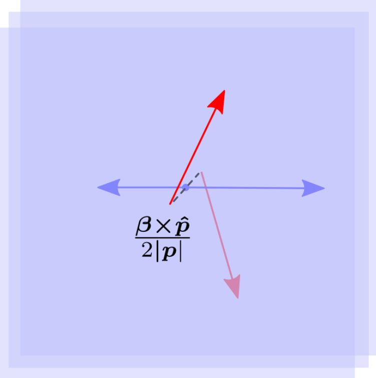

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"Side-jump"

Collision kernel nonlocal

Current

Conservation defines current up to a trivially conserved term (curl).

Liouville current:

$$\boldsymbol{j} = \int_{\boldsymbol{p}} \sqrt{G} f \, \dot{\boldsymbol{x}}$$

Noether current:

$$\boldsymbol{J} \equiv \int_{\boldsymbol{p}} \sqrt{G} f \, \frac{\delta \mathcal{I}}{\delta \boldsymbol{A}} = \boldsymbol{j} + \underbrace{\boldsymbol{\nabla} \times \int_{\boldsymbol{p}} \sqrt{G} f \, \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}}_{\boldsymbol{\nabla} \times \boldsymbol{M}}$$

magnetization current

Lorentz covariant

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magnetization current

Lorentz covariant

Two ways to calculate CVE I: rotating frame

- Replace Lorentz force with Coriolis force (MS, Yin, 2012):

$$\dot{\boldsymbol{p}} = 2\mathcal{E}\,\dot{\boldsymbol{x}} \times \boldsymbol{\omega} \qquad \text{i.e.,} \quad \boxed{\boldsymbol{B} \to 2\mathcal{E}\boldsymbol{\omega}}.$$

- Then CME $\longrightarrow$ CVE

$$\boldsymbol{j}_{\text{CME}} = \boldsymbol{B}\int_{\boldsymbol{p}} f\,\hat{\boldsymbol{p}}\cdot\boldsymbol{b} \quad \longrightarrow \quad \boldsymbol{j}_{\text{CVE}} = \boldsymbol{\omega}\int_{\boldsymbol{p}} 2\mathcal{E}\,f\,\hat{\boldsymbol{p}}\cdot\boldsymbol{b}$$

For example, a distribution $f(\mathcal{E})$ gives

$$\boldsymbol{j}_{\text{CVE}} = \frac{\boldsymbol{\omega}}{4\pi^2}\int_0^\infty f(\mathcal{E})\,2\mathcal{E}\,d\mathcal{E}$$

(cf. Loganayagam-Surowka) and $\dfrac{1}{4\pi^2}\mu^2\boldsymbol{\omega}$ for FD $T = 0$.

NB: B is *not* "$\sim$" $\mu\omega$.

Two ways to calculate CVE II: rotating distribution

For a locally isotropic, but slowly rotating distribution ($\boldsymbol{\nabla} \times \boldsymbol{u} = 2\boldsymbol{\omega}$):

$$f(\mathcal{E}') = f(|\boldsymbol{p}| - \boldsymbol{p}\cdot\boldsymbol{u} - \frac{1}{2}\hat{\boldsymbol{p}}\cdot\boldsymbol{\omega})$$

In the inertial lab frame, the Noether current

$$\boldsymbol{J} = \underbrace{\int_{\boldsymbol{p}} f\hat{\boldsymbol{p}}}_{\text{normal}} + \underbrace{\boldsymbol{\nabla} \times \int_{\boldsymbol{p}} f\frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}}_{\text{magnetization}}$$

equals to

$$\boldsymbol{J} = -\frac{\boldsymbol{\omega}}{2}\left(\frac{1}{3} + \frac{2}{3}\right)\int_{\boldsymbol{p}} \frac{\partial f(\mathcal{E})}{\partial \mathcal{E}} = \frac{\boldsymbol{\omega}}{4\pi^2}\int_0^\infty f\, 2p dp.$$

Must be the same by Lorentz covariance of $\boldsymbol{J}$.

Summary/Conclusions

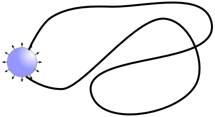
Chiral Kinetic Theory:

- Spin adds $\mathcal{O}(\hbar)$ terms to cl. EOMs: Berry curvature and magnetic mom.

$$\dot{\boldsymbol{x}} = \frac{\partial \mathcal{E}}{\partial \boldsymbol{p}} + \dot{\boldsymbol{p}} \times \boldsymbol{b};$$

$$\mathcal{E} \equiv |\boldsymbol{p}| - \frac{\hat{\boldsymbol{p}} \cdot \boldsymbol{B}}{2|\boldsymbol{p}|}$$

- Berry monopole accounts for CME and anomaly (source/sink at $\boldsymbol{p} = \boldsymbol{0}$).



- Lorentz invariance is realized nontrivially: shift needed to conserve ang. momentum; requires magn. moment coupling

$$\delta \boldsymbol{x} = \boldsymbol{\beta} t + \frac{\boldsymbol{\beta} \times \hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}$$

- CVE from CKT: consistent in rotating frame or rotating distribution

$$\boldsymbol{B} \to 2\mathcal{E}\boldsymbol{\omega}$$

$$\underset{}{\boldsymbol{J}} = \underset{1/3}{\boldsymbol{j}} + \underset{2/3}{\boldsymbol{\nabla} \times \boldsymbol{M}}$$

More . . .

Antiparticles

CP

Same equations with opposite signs of $\boldsymbol{B}$, $\boldsymbol{E}$ and $\boldsymbol{b}$.

The charge current is $\boldsymbol{j} = \boldsymbol{j}_+ - \boldsymbol{j}_-$. The anomaly

$$\partial_\mu j^\mu = (\boldsymbol{E}\cdot\boldsymbol{B})\int_{\boldsymbol{p}} f_+ \boldsymbol{\nabla}_{\boldsymbol{p}}\boldsymbol{b} - \text{(a/p)} = \frac{1}{4\pi^2}\boldsymbol{E}\cdot\boldsymbol{B}\ \underbrace{(f_+ + f_-)_{\boldsymbol{p}=\boldsymbol{0}}}_{=1 \text{ for all } T \text{ and } \mu.}\ .$$

Lorentz algebra

$$\delta \boldsymbol{x} = \boldsymbol{\beta} t + \frac{\boldsymbol{\beta} \times \hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}, \quad \delta \boldsymbol{p} = \boldsymbol{\beta} \mathcal{E} + \frac{\boldsymbol{\beta} \times \hat{\boldsymbol{p}}}{2|\boldsymbol{p}|} \times \boldsymbol{B}, \quad \delta t = \boldsymbol{\beta} \cdot \boldsymbol{x}.$$

Commutator of the ordinary Lorentz transformations is a rotation:

$$\boldsymbol{\varphi} = \boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2 \tag{1}$$

For the modified Lorentz transformation, however,

$$[\delta_{\boldsymbol{\beta}_1}, \delta_{\boldsymbol{\beta}_2}]\boldsymbol{x} = \boldsymbol{\varphi} \times \boldsymbol{x} - \hat{\boldsymbol{p}} \, \frac{\boldsymbol{\varphi} \cdot \hat{\boldsymbol{p}}}{|\boldsymbol{p}|} \tag{2}$$

$$[\delta_{\boldsymbol{\beta}_1}, \delta_{\boldsymbol{\beta}_2}]t = -\frac{\boldsymbol{\varphi} \cdot \hat{\boldsymbol{p}}}{|\boldsymbol{p}|} \tag{3}$$

Reparametrization of the trajectory (shift t): $\delta \boldsymbol{x} = \dot{\boldsymbol{x}} \delta t$.

MM coupling and Heisenberg uncertainty

Path-integral derivation needs to take into account $[x, p] = i$. How?

Discretized path: $\ldots \underbrace{p, x, p'}_{\Delta p = p - p'}, x' \ldots$

$$\int dx\, e^{-ix\Delta p} \underbrace{[F(x)\Delta p]}_{\mathcal{O}(\Delta t)} = \int dx\, e^{-ix\Delta p} \underbrace{[-iF'(x)]}_{\mathcal{O}(1)}$$

If $F(x) = x$ we get

$$\langle px - xp' \rangle = -i.$$