Correspondence of I- and Q-balls as Non-relativistic Condensates

I

Q

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arXiv:1405.3233

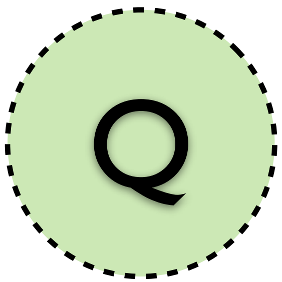
Outline

1. Introduction
2. I-/Q-Ball Correspondence
3. Discussion and Summary

1. Introduction

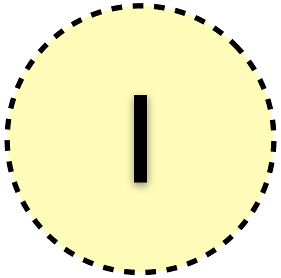
- In scalar field theories, there exist localized objects

- In U(1) charge conserved complex scalar theory



Q-ball S. R. Coleman, 1985

- In a real scalar theory



I-ball(oscillon) M. Gleiser, 1994

1. Introduction

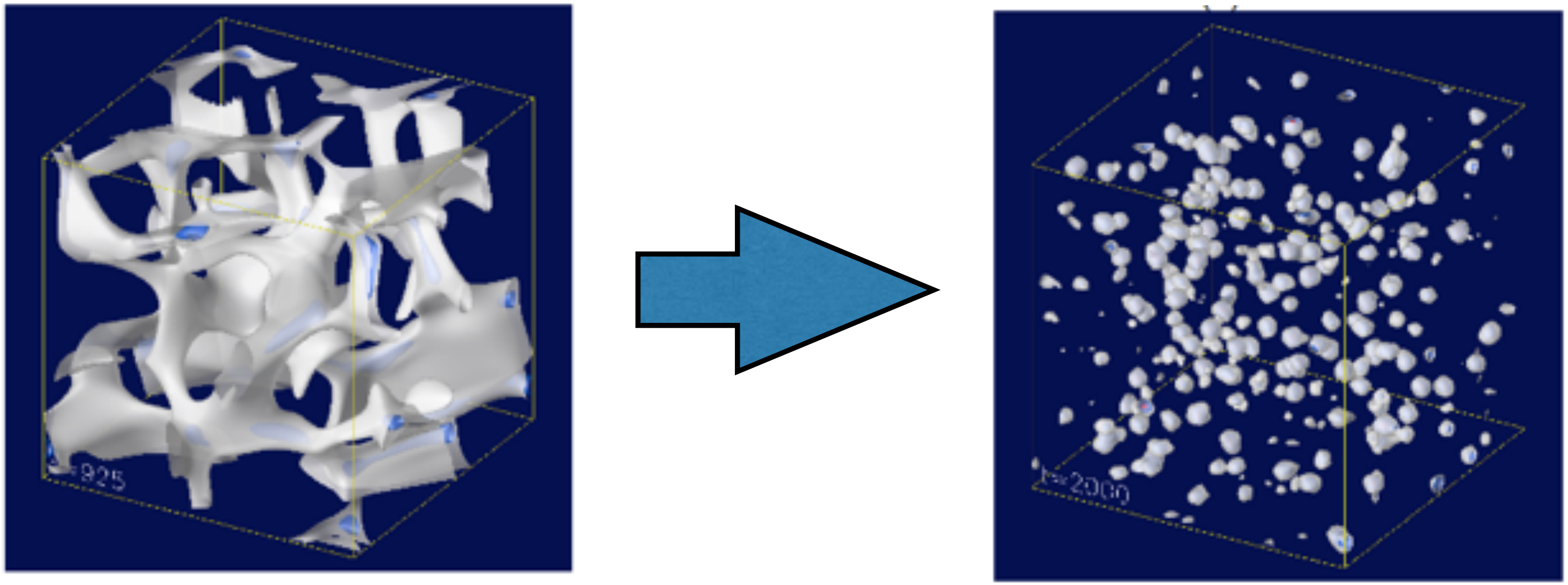
Let us see formations of balls.

With initial conditions which have a little fluctuations, balls are formed in the timescale of oscillation.

Example: Snapshots of Q-ball formation in a 3D lattice simulation

From ``Numerical study of Q-ball formation in gravity mediation'', T. Hiramatsu, M. Kawasaki, F. Takahashi 2010

With fixed initial charge



t=925/m t=2000/m

White surfaces represent regions with high charge density.

One can see balls of charge are formed!!

1. Introduction

Q — I'm stable

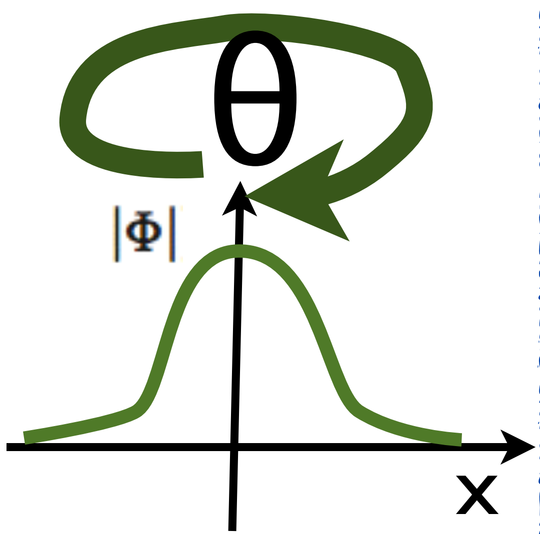
Complex scalar theory
with conserved charge Q

$$\mathscr{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 |\Phi|^2 - V_{\mathrm{U(1)}}(|\Phi|)$$

Φ : complex scalar field

U(1) symmetry: $\Phi \mapsto e^{i\theta}\Phi$

Localized charge



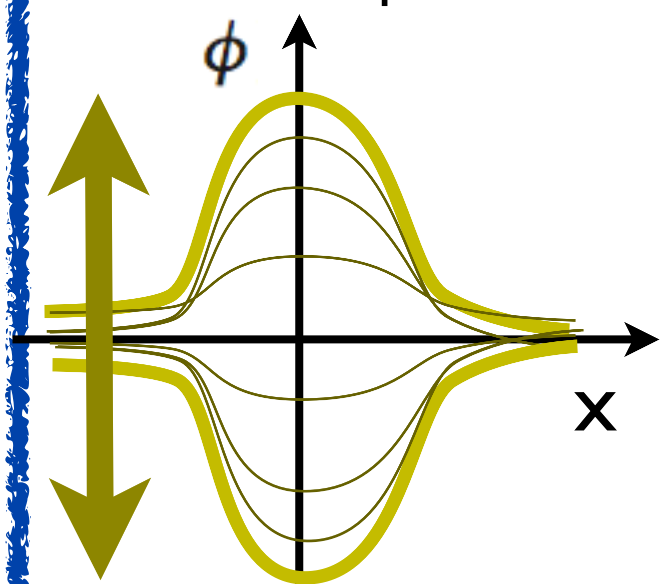
I — I'm quasi stable

Real scalar theory

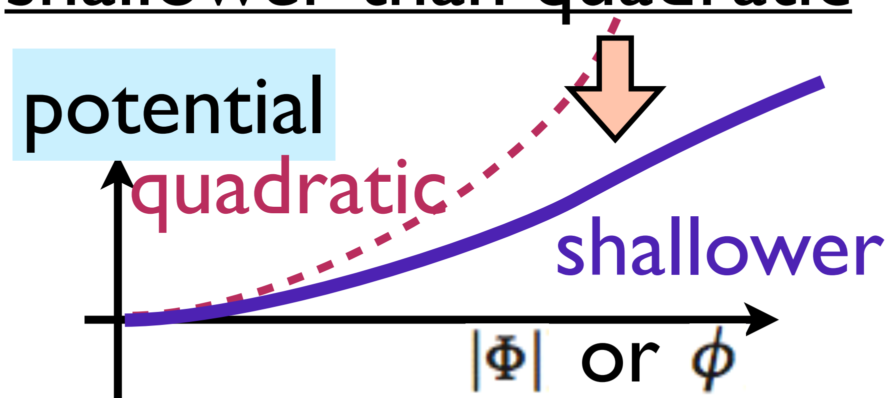
$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi)$$

ϕ : real scalar field

Localized amplitude.



For both cases, balls exist if the potential is a little <u>shallower than quadratic</u>

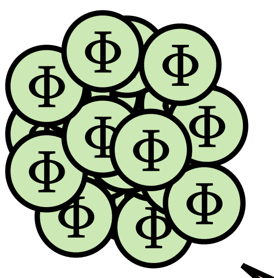


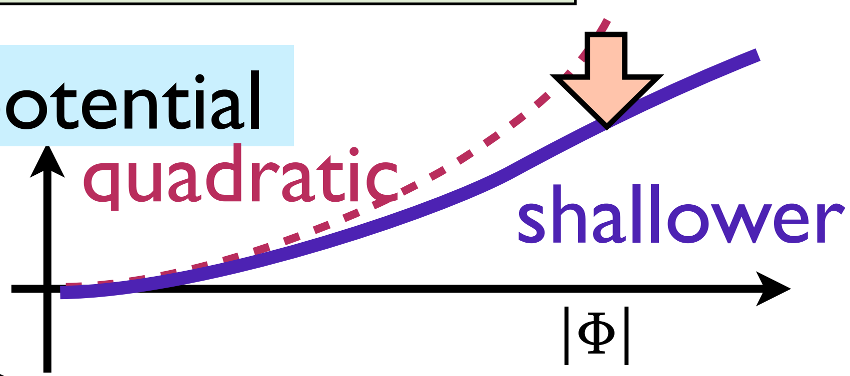
1. Introduction

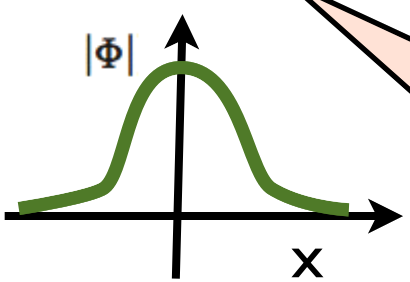
Overview of stability of the Q-ball

Consider the situation with **fixed charge Q** and where non-relativistic modes dominate.

Localized configuration



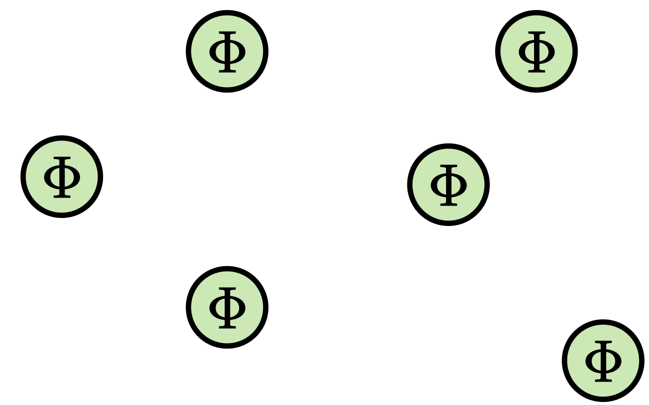




With a large amplitude, effective frequency ω_0 becomes small!!

$$E \simeq \omega_0 Q < mQ$$

Free particles



$$E \simeq mQ$$

Localized configuration is energetically favored!!

1. Introduction

Q

One can show that with fixed charge Q, the Q-ball configuration is energetically most favored.

Stability of the Q-ball is ensured by conserved charge.

I

On the other hand,
the I-ball is an object in a real scalar theory.
There seems no conserved quantity!

Stability of the I-ball seems involved compared to that of Q-ball…

Mysterious!!

1. Introduction

There is a fact:

J. Berges and J. Jaeckel,2014

"Non-relativistic" mode dominated real scalar field obeys E.O.M. of U(1) conserved one!!

Properties of I-ball can be understood by the language of Q-ball!!

Q

Correspondence!!

I

non-relativistic

non-relativistic

Stability of Q-ball is ensured by conserved U(1) charge

Quasi stable!
There seems no conserved quantities

2. Correspondence

Classically, a real scalar field theory can be embedded into a complex one!!

Let's consider a simple example; $\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4$

ϕ: real scalar field

Original E.O.M.

$$(\Box + m^2)\phi + \lambda\phi^3 = 0$$

$$\Re[\Phi] = \phi$$

Introduce a complex scalar field Φ whose real part is ϕ.

$$(\Box + m^2)\Re[\Phi] + \lambda(\Re[\Phi])^3 = 0$$

Note a fact $(\Re[\Phi])^3 = \Re\left[\frac{3}{4}\Phi^2\Phi^\dagger + \frac{1}{4}\Phi^{\dagger 3}\right]$

$$\Re\left[(\Box + m^2)\Phi + \frac{\lambda}{4}\Phi^2\Phi^\dagger + \frac{\lambda}{4}\Phi^{\dagger 3}\right] = 0$$

Construct a complex Lagrangian

$$\mathscr{L} = \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2|\Phi|^2 - V_{\mathrm{U}(1)}(|\Phi|) - V_{\mathrm{B}}(\Phi, \Phi^\dagger)$$

U(1) conserving part

$$V_{\mathrm{U}(1)} = \frac{3\lambda}{8}|\Phi|^4$$

U(1) violating part

$$V_{\mathrm{B}} = \frac{\lambda}{16}\left(\Phi^4 + \Phi^{\dagger 4}\right)$$

2. Correspondence

Embedding is always possible in general!

Original Lagrangian

$$\mathscr{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi)$$

ϕ :real

Φ :complex

$$\Re[\Phi] = \phi$$

Original E.O.M.

$$0 = \left(\Box + m^2\right)\phi + V'(\phi)$$

Find appropriate potentials

Recast into complex one

$$0 = \left(\Box + m^2\right)\Re[\Phi] + V'(\Re[\Phi]) = \Re\left[\left(\Box + m^2\right)\Phi + \frac{\partial V_{\mathrm{U(1)}}(|\Phi|)}{\partial \Phi^\dagger} + \frac{\partial V_{\mathrm{B}}(\Phi, \Phi^\dagger)}{\partial \Phi^\dagger}\right]$$

Complex Lagrangian

$$\mathscr{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 |\Phi|^2 - V_{\mathrm{U(1)}}(|\Phi|) - V_{\mathrm{B}}(\Phi, \Phi^\dagger)$$

$V_{\mathrm{U(1)}}$: U(1) conserving part
V_{B} : U(1) violating part

Generate the same E.O.M. for $\phi = \Re[\Phi]$

2. Correspondence

Examples

Generate the same E.O.M. for $\phi = \Re[\Phi]$

Original interaction potential (log case)

$$V(\phi) = -\frac{1}{2}m^2\phi^2 K \ln\left[\frac{\phi^2}{\phi_0^2}\right] \quad \text{with } K \ll 1$$

Corresponding complex Lagrangian

$$\mathscr{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2|\Phi|^2 - V_{\mathrm{U(1)}}(|\Phi|) - V_{\mathrm{B}}(\Phi, \Phi^\dagger)$$

$$V_{\mathrm{U(1)}} = -m^2|\Phi|^2 K \ln\left[\frac{|\Phi|^2}{\Phi_0^2}\right] \quad \text{with } \Phi^2 = \frac{4}{e}\phi_0^2,$$

$$V_{\mathrm{B}} = \frac{-m^2 K}{6}\frac{\Phi^4 + \Phi^{\dagger 4}}{|\Phi|^2} + \sum_{n\geq 3} c_n m^2 K \frac{\Phi^{2n} + \Phi^{\dagger 2n}}{|\Phi|^{2n-2}} \quad \text{with } c_n \text{ being numerical factors}$$

Then, if the effects from V_{B} are negligible, the dynamics follows the U(1) symmetry!!!

2. Correspondence

If non-relativistic modes dominate, the effects of V_{B} are negligible!!

Two conditions for such a situation (non-relativistic condition)

1. mass term dominates the potential

$$\left|V'_{\mathrm{B}}(\Phi_0, \Phi_0^\dagger)\right| \sim \left|V'_{\mathrm{U(1)}}(|\Phi_0|)\right| \ll m^2|\Phi_0|$$

Φ_0 : typical amplitude

2. Φ can be separated into two part as

$$\Phi = e^{-imt}\Psi + \delta\Phi$$

Ψ : slowly varying field

$$|\Psi| \gg \left|\frac{\partial\Psi}{m\partial t}\right| \gg \left|\frac{\partial^2\Psi}{m^2\partial t^2}\right|$$

$\delta\Phi$: fast oscillating but small

$$|\delta\Phi| \ll |\Psi|$$

Note: Higher momentum modes must be small.

2. Correspondence

E.O.M. for such a configuration can be written as

$$0 = \left[-2im\frac{\partial}{\partial t} - \nabla^2 + U_{\mathrm{U(1)}}(|\Psi|)\right]\Psi; \quad U_{\mathrm{U(1)}}(|\Psi|) \equiv \frac{1}{2|\Psi|}V'_{\mathrm{U(1)}}(|\Psi|),$$

$\Phi = e^{-imt}\Psi + \delta\Phi$

Ψ : slowly varying field

$\delta\Phi$: fast oscillating but small

where we neglect

- higher time derivative on Ψ
- terms containing $\delta\Phi$
- the term $\frac{\partial V_{\mathrm{B}}}{\partial \Phi^\dagger}$ ($\partial V_{\mathrm{B}}/\partial\Phi^\dagger \supset -e^{3imt}(\lambda/4)\Psi^{\dagger 3}$, for $V(\Phi) \supset -\frac{\lambda}{4}\phi^4$)

fast oscillating and averages to zero.
compensated by $\delta\Phi$.

Size of $\delta\Phi$ is suppressed thanks to non-relativistic condition

$$(\Box + m^2)\delta\Phi + V'_{\mathrm{B}} \simeq 0 \longrightarrow |\delta\Phi| \sim |V'_{\mathrm{B}}|/m^2 \ll |\Phi_0|$$

2. Correspondence

We have seen that if the following non-relativistic conditions (1, 2) are satisfied;

1. mass term dominates the potential

$$\left|V'_{\mathrm{B}}(\Phi_0, \Phi_0^\dagger)\right| \sim \left|V'_{\mathrm{U(1)}}(|\Phi_0|)\right| \ll m^2|\Phi_0|$$

Φ_0 : typical amplitude

2. Φ can be separated into two part as

$$\Phi = e^{-imt}\Psi + \delta\Phi$$

Ψ : slowly varying field
$\delta\Phi$: fast oscillating but small

then,

the dynamics is well described by U(1) conserved one!

$$\mathscr{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2|\Phi|^2 - V_{\mathrm{U(1)}}(|\Phi|)$$

Q

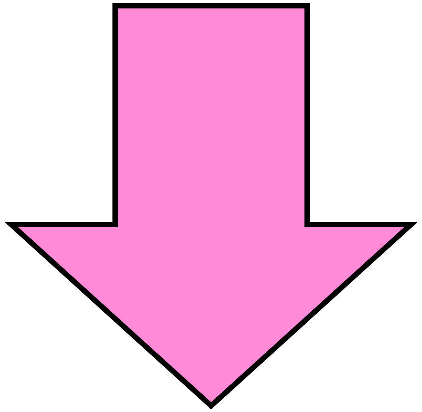
The existence of I-ball in non-relativistic regime can be understood by the language of Q-ball!
I-ball is just a real part of the Q-ball!!!!!

I

3. Discussion

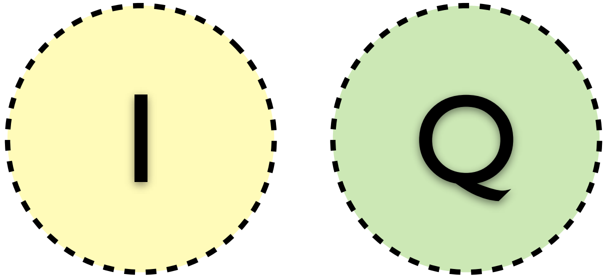
Discussion about the formation of the I-ball.

As long as the non-relativistic condition(1,2) holds,
the dynamics can be described by U(1) conserved one.



The typical timescale of the Q-ball formation is the timescale of oscillation. In that timescale, the effects from $V_{\rm B}$ are supposed to be negligible.

The formation process of the I-ball will be described by that of Q-ball as long as non-relativistic condition holds.



In order to confirm this statement,
we may need a numerical lattice study…

3. Discussion

I — I'm quasi stable

I'm stable — Q

Discussion about the instability of the I-ball.

The shape of the I-ball satisfy the non-relativistic condition.

Balance between pressure and the attractive force: $L^2 V'_{\mathrm{U(1)}}(\Phi_0)/\Phi_0 \sim 1$ with L being a typical saze.

$1/\epsilon \equiv L\omega \simeq Lm \gg 1$ (using $|V'_{\mathrm{U(1)}}(\Phi_0)|/\Phi_0 \gg m^2$) →The shape is wide.

However, the U(1) symmetry is an approximate one.

$$\mathscr{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2|\Phi|^2 - V_{\mathrm{U(1)}}(|\Phi|) - V_{\mathrm{B}}(\Phi, \Phi^\dagger)$$

$V_{\mathrm{U(1)}}$: U(1) conserving part
V_{B} : U(1) violating part

The stability of I-ball is not exactly ensured.

In the paper, we show effects from U(1) violating part are not effective in the time scale of oscillation for some examples. See arXiv:1405.3233 if you are interested in it.

3. Summary

In the non-relativistic regime, a classical real scalar field theory can be embedded into a complex one with a conserved U(1) charge.

From this fact, we have shown an I-ball can be understood as a projection of a Q-ball if the non-relativistic condition holds.

Thank you for listening!!!

Back up!!

Effects from V_B

E. O. M. for $\Phi = \Phi_Q + \delta\Phi$, with Φ_Q being a Q-ball solution;

$$\left(\Box + m^2\right)\delta\Phi = -\left.\frac{\partial V^{(\mathrm{int})}}{\partial \Phi^\dagger}\right|_{\Phi=\Phi_Q+\delta\Phi} + \left.\frac{\partial V_{\mathrm{U}(1)}}{\partial \Phi^\dagger}\right|_{\Phi=\Phi_Q} \quad \text{with } V^{(\mathrm{int})} = V_{\mathrm{U}(1)}(|\Phi|) + V_{\mathrm{B}}(\Phi, \Phi^\dagger).$$

We divide $\delta\Phi$ into two parts;

$\delta\Phi = \delta\Phi_{\mathrm{cmp}} + \delta\Phi_{\mathrm{flc}}$ where $\delta\Phi_{\mathrm{cmp}}$ compensates V'_{B} and $\delta\Phi_{\mathrm{flc}}$ is fluctuations around the ball.

For quartic potentials;

$$\left(\Box + m^2 - \frac{3\lambda}{2}|\Phi_Q|^2\right)\delta\Phi_{\mathrm{cmp}} \simeq \frac{\lambda}{4}\Phi_Q^{\dagger\,3}$$ →Just oscillating.

$$\left(\Box + m^2 - \frac{3\lambda}{2}|\Phi_Q|^2\right)\delta\Phi_{\mathrm{flc}} = \frac{3\lambda}{4}\left(\Phi_Q^2 + \Phi_Q^{\dagger\,2}\right)\delta\Phi_{\mathrm{flc}}^\dagger$$ →particle productions may happen.

Back up!!

Particle productions

Example; quartic and sixtic potential

$$V_{\mathrm{U}(1)}(|\Phi|) = -\frac{3\lambda}{8}|\Phi|^4 + \frac{5g}{24m^2}|\Phi|^6,$$

$$V_{\mathrm{B}}(\Phi, \Phi^\dagger) = -\frac{\lambda}{16}\left(\Phi^4 + \Phi^{\dagger 4}\right) + \frac{g|\Phi|^2}{16m^2}\left(\Phi^4 + \Phi^{\dagger 4}\right)$$

Four to two

$$\Gamma_{I,4\to 2} \sim \begin{cases} \lambda\epsilon^6\omega & \text{by quartic interaction} \\ \lambda\epsilon^6\left(\frac{g}{\lambda^2}\right)^2\omega \lesssim \lambda\epsilon^2\omega & \text{by sixtic interaction} \end{cases},$$

$\epsilon \equiv 1/\omega L \ll 1$ (L: typical size of the ball)

Back up!!

Two to two charge violating process may happen.

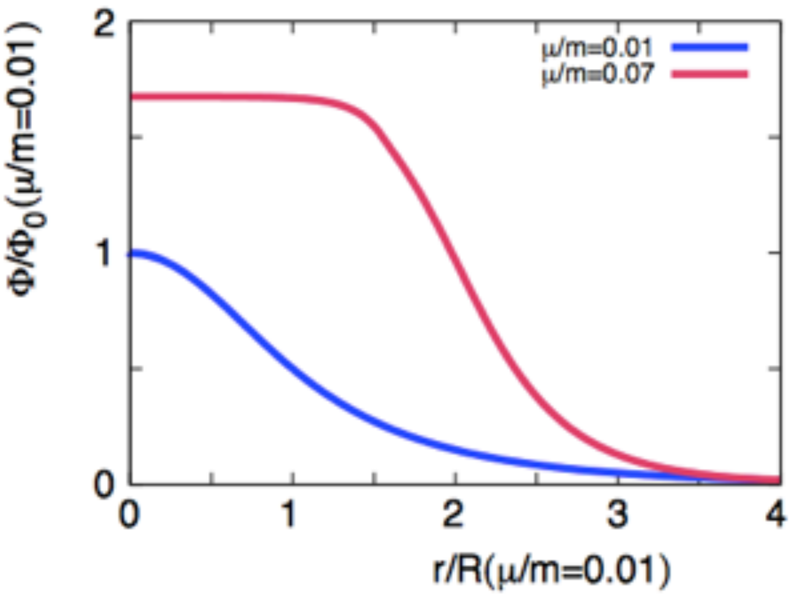
Assume there exist a mode with frequency omega inside the ball.

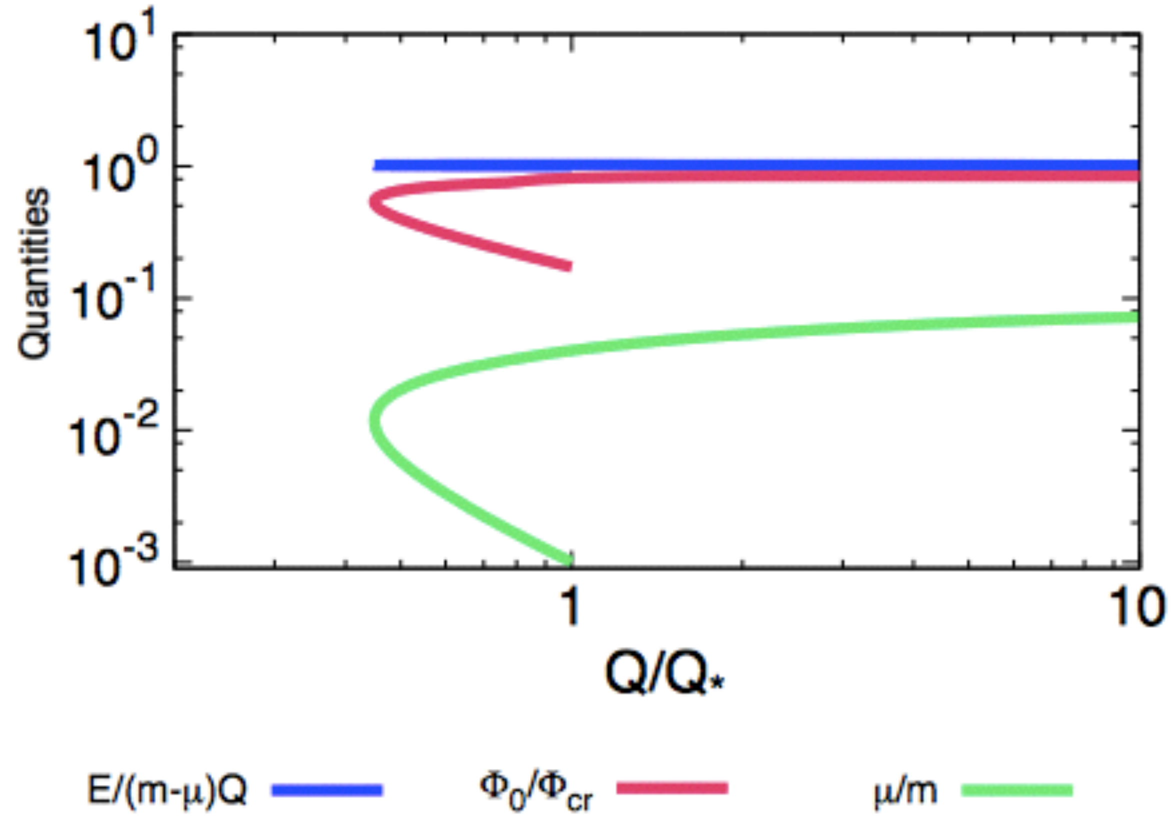
$$\begin{aligned}\bar{E}' &\simeq \omega(Q-\Delta Q)\cdot[Q-\Delta Q]+\omega(Q)\Delta Q\\ &\simeq \omega(Q)Q-\frac{\partial\omega(Q)}{\partial Q}Q\Delta Q\\ &\geq \omega(Q)Q,\end{aligned}$$

The critical condition

$$\frac{\partial\omega(Q)}{\partial Q}<0,$$

Back up!!





$$V_{\mathrm{U(1)}}(|\Phi|) = -\frac{3\lambda}{8}|\Phi|^4 + \frac{5g}{24m^2}|\Phi|^6.$$

$$\omega = m - \mu$$

There are two regime: $\partial\omega/\partial Q \lessgtr 0.$

Back up!!

Examples

Original potential

$$V(\phi) = -\frac{\lambda}{4}\phi^4 + \frac{g}{6m^2}\phi^6$$

Generate the same E.O.M. for $\phi = \Re[\Phi]$

Corresponding complex potential

$$V_{\mathrm{U(1)}}(|\Phi|) = -\frac{3\lambda}{8}|\Phi|^4 + \frac{5g}{24m^2}|\Phi|^6,$$

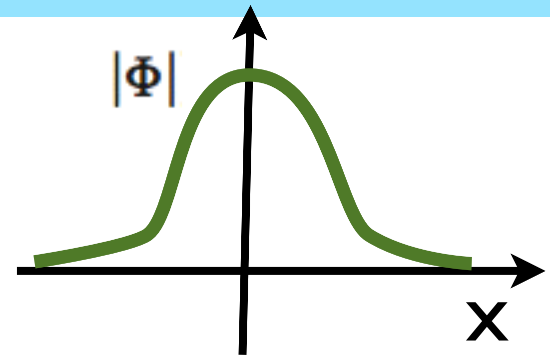
$$V_{\mathrm{B}}(\Phi, \Phi^\dagger) = -\frac{\lambda}{16}\left(\Phi^4 + \Phi^{\dagger 4}\right) + \frac{g|\Phi|^2}{16m^2}\left(\Phi^4 + \Phi^{\dagger 4}\right)$$

Then, if the effects from V_{B} is negligible, the dynamics follows the U(1) symmetry!!!

Back up!!

Q

More strictly speaking, stability of the Q-ball can be seen as following.



Conserved charge $Q = i \int (\Phi^\dagger \partial_0 \Phi - \Phi \partial_0 \Phi^\dagger) d^3x$

With fixed Q, the lowest energy configuration can be found by the method of Lagrange multiplier

$$\Gamma[\Phi, \omega_0] \equiv E + \omega_0 \left[Q - i \int (\Phi^\dagger \partial_0 \Phi - \Phi \partial_0 \Phi^\dagger) d^3x \right]$$ with $E = \int (\partial_0 \Phi^\dagger \partial_0 \Phi + \partial_i \Phi^\dagger \partial_i \Phi + m^2 \Phi^\dagger \Phi + V_{\mathrm{U(1)}}) d^3x$

The condition $\dfrac{\delta \Gamma[\Phi, \omega_0]}{\delta \Phi} = 0$ may have a bounce solution

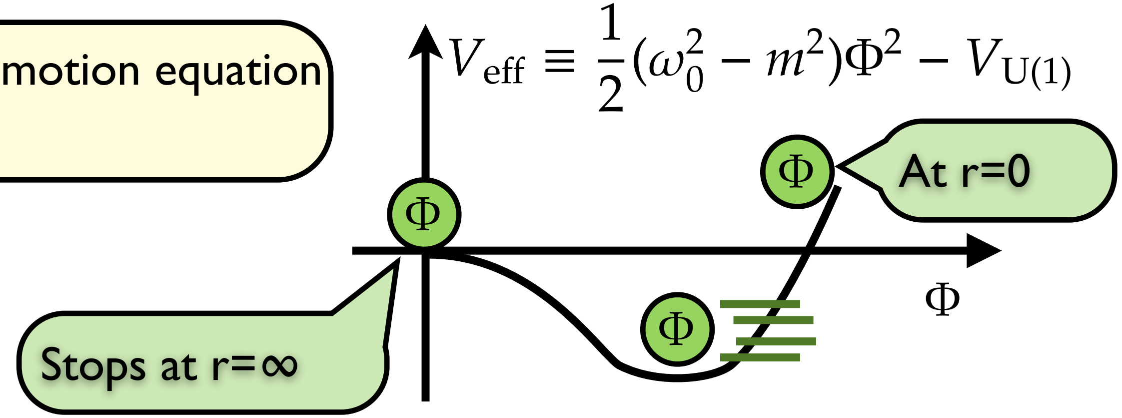
$$\Phi(x) = \Phi(r) e^{-i\omega_0 t}$$

$$\frac{\partial^2}{\partial r^2}\Phi + \frac{2}{r}\frac{\partial}{\partial r}\Phi + (\omega_0^2 - m^2)\Phi - \frac{1}{2}\frac{\partial V_{\mathrm{U(1)}}}{\partial \Phi} = 0$$

$$\frac{\partial \Phi(r=0)}{\partial r} = \Phi(r \to \infty) = 0$$

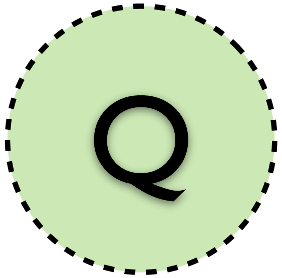
Just like a Newton's motion equation with a friction.

$$V_{\mathrm{eff}} \equiv \frac{1}{2}(\omega_0^2 - m^2)\Phi^2 - V_{\mathrm{U(1)}}$$



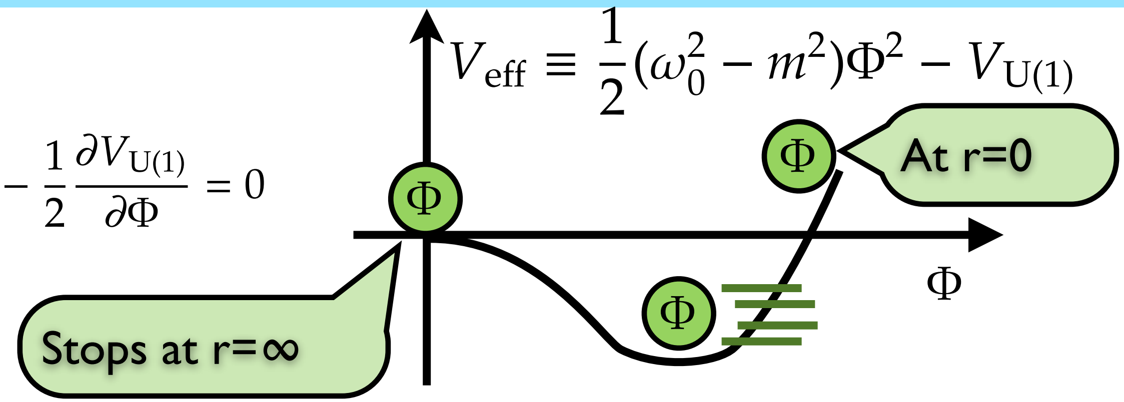
With fixed Q, the lowest energy configuration becomes localized one.

Back up!!

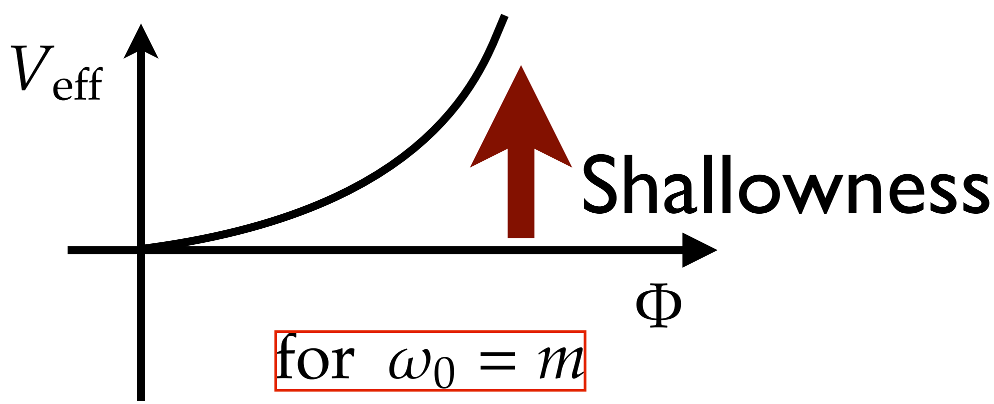


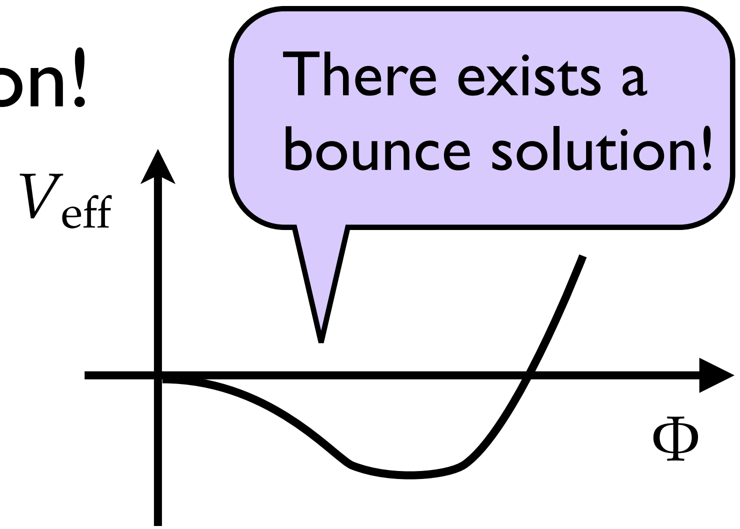
$$\frac{\partial^2}{\partial r^2}\Phi + \frac{2}{r}\frac{\partial}{\partial r}\Phi + (\omega_0^2 - m^2)\Phi - \frac{1}{2}\frac{\partial V_{U(1)}}{\partial \Phi} = 0$$

$$\Phi(x) = \Phi(r)e^{-i\omega_0 t}$$



The shallowness of the potential
is important to have a bounce solution!





for $\omega_0 = m - \mu,\ \mu \ll m$

Note that $E \simeq \omega_0 Q < mQ$

With a large amplitude, the effective mass ω_0 becomes small.

Well understandable!!

Back up!!

Typical size of the balls

Balance between pressure and the attractive force:

$L^2 V'_{U(1)}(\Phi_0)/\Phi_0 \sim 1.$ L: typical size of the ball.

Non-relativistic condition

$\left|V_B(\Phi_0, \Phi_0^\dagger)\right| \sim \left|V_{U(1)}(|\Phi_0|)\right| \ll m^2|\Phi_0|$

$1/\epsilon \equiv L\omega \simeq Lm \gg 1,$

Typical size is large!!

Back up!!

Embedding is always possible in general!!

We consider a real scalar ϕ field theory with the potential $V(2\phi^2)$. The equation of motion is the following

$$\Box\phi + 4V'(2\phi^2)\phi = 0. \tag{A.1}$$

We can obtain the corresponding complex Lagrangian;

$$\mathscr{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2|\Phi|^2 - V_{\mathrm{U(1)}}(|\Phi|) - V_{\mathrm{B}}(\Phi, \Phi^\dagger)$$

with

$$V_{\mathrm{U(1)}}(|\Phi|^2) \equiv G_0(|\Phi|^2),$$
$$V_{\mathrm{B}}(\Phi, \Phi^\dagger) \equiv \sum_{n\geq 2} G_n(|\Phi|^2)\left[\Phi^{2n} + \Phi^{\dagger 2n}\right]$$

$$2(n+1)G_{n+1}(x) + xG'_{n+1}(x) = 2g_n(x) - G'_n(x). \quad , G_1(x) = 0$$

$$g_n(r^2) = \frac{2}{\pi}\int_{-\pi}^{\pi} \frac{V'\left(r^2(1+\cos 2\theta)\right)\cdot\cos\theta\cdot\cos((2n+1)\theta)}{r^{2n}}d\theta$$

Back up!!

For example, in the case of polynomial potential $V(x) = Ax^{m+1}/(m+1)$ with constant A, $G_n(x)$ can be obtained as

$$G_n(x) = 2^{2-m} A \frac{1+(-1)^n}{2} \frac{{}_{2m+1}\mathrm{C}_{m-n+1}}{m+n+1} x^{m+1-n} \text{ for } n \leq m+1 \text{ otherwise zero.} \tag{A.15}$$

$$V_{\mathrm{U}(1)}(|\Phi|^2) \equiv G_0(|\Phi|^2),$$

$$V_{\mathrm{B}}(\Phi, \Phi^\dagger) \equiv \sum_{n \geq 2} G_n(|\Phi|^2) \left[\Phi^{2n} + \Phi^{\dagger 2n} \right]$$