Worldline models of lattice gauge theories beyond the strong coupling limit

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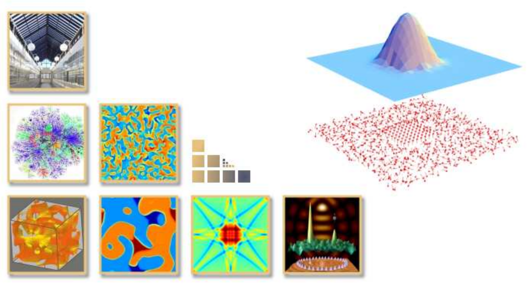

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1 Introduction

Lattice gauge theories describing QCD-like systems **at finite temperature** T **and density** μ, e.g. N_f flavours of massive staggered fermions $\chi^\alpha_x, \bar{\chi}^\alpha_x$ coupled to $U(N)$ or $SU(N)$ gauge fields $U_{x,\mu}$ with partition function:

$$Z = \int [d\chi d\bar{\chi}] \int [dU]\, e^{-S_G - S_F}$$

with the gauge and the fermion actions given by:

$$S_G = \beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{N}\mathrm{ReTr}(U_{x,\mu}U_{x+\hat{\mu},\nu}U^\dagger_{x+\hat{\nu},\mu}U^\dagger_{x,\nu})\right)$$
$$S_F = \sum_{x,\mu,\alpha} \eta_{x\mu}(e^{\mu_\alpha a_\tau \delta_{\mu\tau}}\bar{\chi}^\alpha_x U_{x,\mu}\chi^\alpha_{x+\hat{\mu}} - e^{-\mu_\alpha a_\tau \delta_{\mu\tau}}\bar{\chi}^\alpha_{x+\hat{\mu}}U^\dagger_{x,\mu}\chi^\alpha_x) + \sum_{x,\alpha} am_\alpha \bar{\chi}^\alpha_x \chi^\alpha_x$$

are plagued by the **sign problem** once the fermions are integrated out. This prevents direct numerical simulation based on importance sampling. A proposed way to ameliorate this problem consists in integrating out the gauge fields *before* the fermion fields. Such is known to be possible only for $\beta = 0 \Leftrightarrow S_G = 0$, which reduces Z to a product of solvable fermionic one-link integrals.

Integrating out the remaining fermions, e.g. for $N_f = 1$, results in a **monomer-dimer-polymer (MDP) model** describing the ensemble of **worldlines of free color singlets** [1, 2]:

$$Z = \sum_{\{n,k,\ell\}} w_M(n)\, w_D(k)\, w_P(\ell)$$
$$w_M(n) = \prod_x \frac{N!}{n_x!}(am)^{n_x}, \qquad w_D(k) = \prod_l \frac{(N-k_l)!}{N!k_l!}$$
$$w_P(\ell) = \sigma(\ell)\, e^{r(\ell)\mu/T}, \qquad \sigma(\ell) = (-1)^{r(\ell)+N_-(\ell)+1}\prod_{l\in\ell}\eta_l$$

where n_x and k_l are monomer and dimer occupation numbers describing quark condensates and meson hopping, respectively, and ℓ are non-self-intersecting loops (polymers) winding $r(\ell)$ times around the Euclidean time direction, which describe baryon hopping; $\sigma(\ell) = \pm 1$ is a residual sign, which can be tamed numerically.

Recent numerical simulations [3, 4, 5] successfully mapped the phase diagram of $N_f = 1$ $SU(3)$ QCD in the T-μ plane near $\beta = 0$. The problem remains, however, on how to approach the regime of continuum physics ($\beta \to \infty$) in this worldline approach.

2 Integrating the gauge fields

We address the problem of how to **integrate out the gauge fields exactly** in the pure gauge theory at $\beta \neq 0$:

$$Z_4 = \int [dU]\, e^{-\beta\sum_{x,\mu<\nu}\left(1-\frac{1}{N}\mathrm{ReTr}(U_{x,\mu}U_{x+\hat{\mu},\nu}U^\dagger_{x+\hat{\nu},\mu}U^\dagger_{x,\nu})\right)}$$

We multiply the original partition function by '1':

$$Z_4 \equiv Z_4 \times \int \gamma_1[Q'] \int \gamma_1[R']$$

in the form of normalized Gaussian integrals over auxiliary $\mathrm{Mat}(N,\mathbb{C})$ fields $Q'_{x,\mu\nu}, R'_{x,\mu\nu}$ living on plaquettes (Fig.1). The $\gamma_a(z) = \frac{a}{2\pi}\, dz^* dz\, e^{-\frac{a}{2}|z|^2}$ are normalized Gaussian measures.

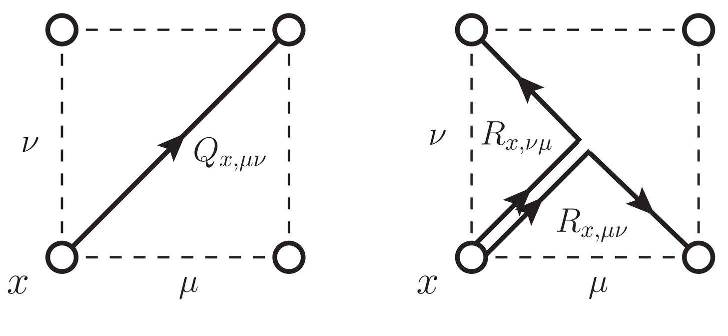

Fig. 1: *Auxiliary Gaussian variables: the diagonal link (left) splits the original plaquette into two halves, and the folded links (right) split each half into two quarters.*

By performing a sequence of **Hubbard-Stratonovich (HS) transformations** on the auxiliary fields:

$$Q_{x,\mu\nu} = \sqrt{\frac{N}{\beta}}Q'_{x,\mu\nu} + U_{x,\mu}U_{x+\hat{\mu},\nu} + U_{x,\nu}U_{x+\hat{\nu},\mu}$$
$$R_{x,\mu\nu} = \sqrt{\frac{N}{\beta}}R'_{x,\mu\nu} + Q_{x,\mu\nu}U^\dagger_{x+\hat{\mu},\nu} + U_{x,\mu}$$

we obtain a sequence of what we call n**-link theories**, Z_n, whose actions are n-linear with respect to the link variables:

$$Z_2 = \int \gamma_{\frac{\beta}{N}}[Q] \int [dU]\, e^{-\beta\sum_{x,\mu<\nu}\left(1-\frac{1}{N}\mathrm{ReTr}(Q^\dagger_{x,\mu\nu}U_{x,\mu}U_{x+\hat{\mu},\nu})\right)}$$
$$Z_1 = \int \gamma_{\frac{3\beta}{N}}[Q]\gamma_{\frac{\beta}{N}}[R]\prod_{x,\mu}\int dU\, e^{\frac{\beta}{N}\mathrm{ReTr}(J^\dagger_{x,\mu}U)}$$
$$Z_0 = \int \gamma_{\frac{3\beta}{N}}[Q]\gamma_{\frac{\beta}{N}}[R]\prod_{x,\mu}\mathcal{I}\left(\frac{\beta}{2N}J_{x,\mu}, \frac{\beta}{2N}J^\dagger_{x,\mu}\right)$$

where $J_{x,\mu}$ is the effective sum of staples around the link (x,μ):

$$J_{x,\mu} = \sum_{\nu\neq\mu}(R^\dagger_{x-\hat{\nu},\nu\mu}Q_{x-\hat{\nu},\mu\nu} + R_{x,\mu\nu})$$

and $\mathcal{I}(A,B) = \int dU \exp(\mathrm{Tr}(AU^\dagger) + \mathrm{Tr}(BU))$ are **one-link integrals** over the gauge group [6].

For example, the 0-link partition functions for $U(1)$ and $SU(2)$ pure lattice gauge theories are, respectively:

$$Z_0 = \int \gamma_{3\beta}[Q]\gamma_\beta[R]\prod_{x,\mu} I_0(\beta|J_{x,\mu}|)$$
$$Z_0 = \int \gamma_{\frac{3\beta}{2}}[Q]\gamma_{\frac{\beta}{2}}[R]\prod_{x,\mu}\frac{2I_1(\frac{\beta}{2}z_{x,\mu})}{\frac{\beta}{2}z_{x,\mu}}$$

where $z_{x,\mu} = \sqrt{\mathrm{Tr}(JJ^\dagger)_{x,\mu} + \det(J_{x,\mu}) + \det(J^\dagger_{x,\mu})}$ is an $SU(2)$ invariant, and $I_\nu(z)$ are modified Bessel functions of the first kind.

2.1 Symmetries and observables

All n-link actions share the same **gauge symmetry** (left) and global **center symmetry** (right):

$$U_{x,\mu} \mapsto \Omega_x\, U_{x,\mu}\, \Omega^\dagger_{x+\hat{\mu}} \qquad U_{x,\mu} \mapsto e^{i\theta}\, U_{x,\mu}$$
$$Q_{x,\mu\nu} \mapsto \Omega_x\, Q_{x,\mu\nu}\, \Omega^\dagger_{x+\hat{\mu}+\hat{\nu}} \qquad Q_{x,\mu\nu} \mapsto e^{i2\theta}\, Q_{x,\mu\nu}$$
$$R_{x,\mu\nu} \mapsto \Omega_x\, R_{x,\mu\nu}\, \Omega^\dagger_{x+\hat{\mu}} \qquad R_{x,\mu\nu} \mapsto e^{i\theta}\, R_{x,\mu\nu}$$
$$J_{x,\mu} \mapsto \Omega_x\, J_{x,\mu}\, \Omega^\dagger_{x+\hat{\mu}} \qquad J_{x,\mu} \mapsto e^{i\theta}\, J_{x,\mu}$$

Gauge-invariant operators like non-self-intersecting Wilson loops:

$$W(C) = \frac{1}{N}\mathrm{Tr}\left(\mathcal{P}\prod_{l\in C}U_l\right) = \frac{1}{N}\mathrm{Tr}\left(\mathcal{P}\prod_{l\in C}\widetilde{U}_l\right)$$

may be constructed out of ordinary link variables U_l ($n > 0$), or effective links $\widetilde{U}_l$ ($n = 0$), defined by:

$$\widetilde{U}^{ij}_l = \langle U^{ij}\rangle = \frac{2N}{\beta}\frac{\partial}{\partial(J^\dagger_l)^{ji}}\log\mathcal{I}\left(\frac{\beta}{2N}J_l, \frac{\beta}{2N}J^\dagger_l\right)$$

2.2 Monte Carlo simulations

Link variables and auxiliary variables are treated on an equal footing when it comes to local updates. Auxiliary variables are updated via a Gaussian heatbath followed by HS transformations; link variables are updated via the Cabibbo-Marinari algorithm.

The numerical simulations of each n-link action return the same expectation values for the plaquette, as expected (Fig.2).

	$U(1)$		$SU(2)$	
	β	plaquette	β	plaquette
Z_4	1.00	0.58570(23)	2.25	0.586189(37)
Z_2	1.00	0.58572(59)	2.25	0.586161(65)
Z_1	1.00	0.5864(11)	2.25	0.58618(12)
Z_0	1.00	0.5864(11)	2.25	0.58618(12)

Fig. 2: *Expectation values of the plaquette operator in numerical simulations of n-link actions on a 8^4 lattice.*

3 MDP models at finite β

We extend the 0-link theory to include N_f flavours of staggered fermions, by generalizing the one-link integrals with sources of the form:

$$\frac{\beta}{2N}J^{ij}_{x,\mu} \mapsto \frac{\beta}{2N}J^{ij}_{x,\mu} + \eta_{x\mu}\sum_{\alpha=1}^{N_f} e^{+\mu_\alpha a_\tau\delta_{\mu\tau}}\chi^{\alpha i}_x\bar{\chi}^{\alpha j}_{x+\hat{\mu}}$$
$$\frac{\beta}{2N}\bar{J}^{ij}_{x,\mu} \mapsto \frac{\beta}{2N}\bar{J}^{ij}_{x,\mu} - \eta_{x\mu}\sum_{\alpha=1}^{N_f} e^{-\mu_\alpha a_\tau\delta_{\mu\tau}}\chi^{\alpha i}_{x+\hat{\mu}}\bar{\chi}^{\alpha j}_x$$

This is possible because S_F is linear with respect to the link variables. In the case of $U(1)$ lattice gauge theory with N_f flavours, after integrating out the link variables and fermion fields, we obtain a $U(1)$ **MDP model for arbitrary values of** β:

$$Z(\beta) = \sum_{\{n,k,\ell\}} w_G(\beta,\Delta q)\prod_{\alpha=1}^{N_f}w_M(n_\alpha)\prod_{\alpha,\beta=1}^{N_f}w_D(k_{\alpha\beta})\prod_{\alpha=1}^{N_f}w_P(\ell_\alpha)$$
$$w_G(\beta;\Delta q) = \int\gamma_{3\beta}[Q]\gamma_\beta[R]\prod_l I_{|\Delta q_l|}(\beta|J_l|)\left(\frac{J^\dagger_l}{|J_l|}\right)^{\Delta q_l}$$
$$w_M(n_\alpha) = \prod_x\frac{N!}{n^\alpha_x!}(am_\alpha)^{n^\alpha_x}, \quad w_D(k_{\alpha\beta}) = \prod_l\frac{(N-k^{\alpha\beta}_l)!}{N!k^{\alpha\beta}_l!} \equiv 1$$
$$w_P(\ell_\alpha) = \sigma(\ell_\alpha)\, e^{r(\ell_\alpha)\mu_\alpha/T}$$

which sums over the occupation numbers of flavoured monomers (n^α_x:●,●,...) and oriented electron dimers (q^α_l:⟶,⟶,...), which are constrained to form closed electron loops (ℓ_α). A new gauge contribution, $w_G(\beta,\Delta q)$, is sourced by local forward-backward imbalances $\Delta q_l = \sum_\alpha(q^\alpha_l - \bar{q}^\alpha_l)$. For $N_f = 1$, w_G is the expectation value of the Wilson loop along electron worldlines (Fig.3).

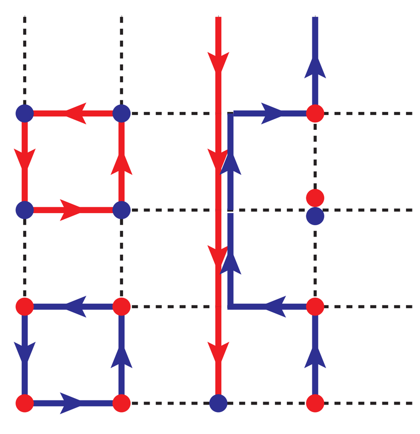

Fig. 3: *Configuration of the MDP model of $N_f = 2$ QED on a 4^2 lattice. Fermion-gauge interactions are induced by the local forward-backward imbalances, Δq_l, that source the gauge contribution, $w_G(\beta;\Delta q)$.*

The extension of the $U(1)$ MDP model to non-Abelian gauge groups, e.g. $SU(2)$, follows the same reasoning. The integrand in the gauge contribution $w_G(\beta;\Delta q)$ becomes different in that case, and extra weights appear in the partition function, which describe quark flow between color singlets.

4 Acknowledgements

HV was supported by FCT (Portugal) through the grants SFRH/BPD/37949/2007 and CERN/FP/123599/2011.

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